

The Steep Nekhoroshev's Theorem

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Abstract

Revising Nekhoroshev's geometry of resonances, we provide a fully constructive and quantitative proof of Nekhoroshev's theorem for steep Hamiltonian systems proving, in particular, that the exponential stability exponent can be taken to be $1/(2n\alpha_1 \cdots \alpha_{n-2})$ (α_i 's being Nekhoroshev's steepness indices and $n \geq 3$ the number of degrees of freedom).

1 Introduction and results

A. Motivations. In 1977-1979 N.N. Nekhoroshev published a fundamental theorem ([19, 20]) about the “exponential stability” (i.e., “stability of action variables over times exponentially long with the inverse of the perturbation size”) of nearly-integrable, real-analytic Hamiltonian systems with Hamiltonian given, in standard action-angle coordinates, by

$$H(I, \varphi) = h(I) + \varepsilon f(I, \varphi), \quad (I, \varphi) \in U \times \mathbb{T}^n, \quad (1)$$

where: $U \subseteq \mathbb{R}^n$ is an open region, $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ is the standard flat n -dimensional torus and ε is a small parameter. The integrable limit $h(I)$ is assumed to satisfy a geometric condition, called by Nekhoroshev “steepness” (the definition is recalled in (3) below). Under such assumptions, Nekhoroshev's states his theorem as follows¹:

Let H in (1) be real-analytic with h steep. Then, there exist positive constants a , b and ε_0 such that for any $0 \leq \varepsilon < \varepsilon_0$ the solution (I_t, φ_t) of the (standard) Hamilton equations for $H(I, \varphi)$ satisfies

$$|I_t - I_0| \leq \varepsilon^b$$

¹Compare [19, p. 4 and p. 8]; see also [19, p. 30] for a more detailed and precise statement.

for any time t satisfying

$$|t| \leq \frac{1}{\varepsilon} \exp\left(\frac{1}{\varepsilon^a}\right).$$

Furthermore, a and b can be taken as follows:

$$a = \frac{2}{12\zeta + 3n + 14}, \quad b = \frac{3a}{2\alpha_{n-1}} \quad (2)$$

where

$$\zeta = \left[\alpha_1 \cdot \left(\alpha_2 \left(\dots \left(\alpha_{n-3} (n\alpha_{n-2} + n - 2) + n - 3 \right) + \dots \right) + 2 \right) + 1 \right] - 1,$$

and α_i are the steepness indices of h .

Usually, a and b are called the “stability exponents”. Clearly, the most relevant quantity in this theorem is the stability exponent a appearing in the exponential, which gives the dominant time-scale for the stability of the action variables. The exponential stability exponent a depends only on the number n of degrees of freedom and on the values of the first $n-2$ steepness indices α_i , $i \leq n-2$. Notice that, for any fixed n , the “best” exponents a, b in (2) are obtained in the special case $\alpha_1 = \dots = \alpha_{n-1} = 1$, corresponding to convex (or quasi-convex) $h(I)$ (which is the simplest instance of steep function). Actually, for any values of the steepness indices α_i , the parameter ζ defined in (3) grows faster than² $n(n-1)/2$. The hypotheses of Nekhoroshev’s theorem, as pointed out by Nekhoroshev himself, are qualitatively optimal, and, in particular, non-steep Hamiltonian are in general non exponentially-stable [19, §11]. Furthermore, Nekhoroshev proved that steepness is a generic (in C^∞ category) property [18]. Finally, several interesting problems (e.g., in Celestial Mechanics, compare below) are steep but do not satisfy simpler assumptions (such as quasi-convexity). For all these reasons it seems natural and important to try to optimize the exponential stability exponents, especially with respect to the number n of the degrees of freedom which, in applications, typically range from $n = 3$ (restricted three-body problems) up to several tens (planetary problems); this has been done, up to now, under simplifying assumptions but not in the general steep case. This paper is devoted to the general case.

Before stating our result, let us briefly review the main extensions, applications and improvements concerning Nekhoroshev’s theorem.

Various extensions have been discussed, so as to cover the degeneracies of the Hamilton function which are usually met in some important mechanical systems (fast rotations of the Euler–Poincot rigid body [1, 2, 3]; the planetary N -body problem [19, 21, 9]; restricted three body problems [8], elliptic equilibria [10, 22, 13, 28]). Furthermore, steepness could be used, in non-convex systems, to study the long-term stability in problems such as the Lagrangian equilibrium points L4-L5 of the restricted three body problem [4], asteroids of the Main Belt [17, 26, 14] and the Solar System [29].

²For any fixed sequence $\alpha_j \geq 1$, $j = 1, 2, \dots$, by considering a sequence of steep Hamiltonians h_n with n degrees of freedom and steepness indices $\alpha_1, \dots, \alpha_{n-1}$, the sequence of corresponding parameters $\zeta_n := \zeta$ satisfies $\zeta_n - \zeta_{n-1} \geq (n-1)\alpha_1 \cdots \alpha_{n-2}$, and the sequence of stability exponents $a_n := a$ satisfies $a_n^{-1} - a_{n-1}^{-1} \geq 6(n-1)\alpha_1 \cdots \alpha_{n-2}$.

As far as improvements of the theoretical stability bounds (i.e., improvements on the stability exponent a), quite complete results have been achieved in the special case of convex and quasi-convex functions h : the proof of the theorem has been significantly simplified (see [11, 5, 6]) and the stability exponent improved up to $a = (2n)^{-1}$, ([16], [15], [27]; see [7] for exponents which are intermediate between $a = (2n)^{-1}$ and $a = (2(n-1))^{-1}$): such exponents (in the convex case) are nearly optimal, compare [30]. These improvements have been obtained by exploiting specific geometric properties of the convex and quasi-convex cases, which allow to use conservation of energy in order to obtain topological confinement of the actions ([6]). In fact, in the convex case, the analysis of the geometry of resonances, that is, the geometry of the manifolds

$$\{I \in U : k \cdot \omega(I) = 0\}, \quad \text{with } \omega(I) = \nabla h(I) \text{ and } k \in \mathbb{Z}^n,$$

is greatly simplified, since the frequency map $I \mapsto \omega(I)$ is a diffeomorphism; on the other hand, in the general steep case, the Hamilton function cannot be used anymore in order to obtain topological confinement, and the geometry of resonances is significantly more complicate, due to possible folds and other degeneracies of the frequency map. Furthermore, while new different proofs of Nekhoroshev's theorem have appeared (compare [24], which is based on the method of simultaneous Diophantine approximations introduced in [15]), *no improvements on the original Nekhoroshev's stability exponents, in the general steep case, are yet available*³.

In this paper, we revisit and extend Nekhoroshev's geometric analysis obtaining, in particular, for⁴ $n \geq 3$, the *new stability exponents* $a = 1/(2np_1)$ and $b = a/\alpha_{n-1}$ with p_1 being the product of the first $(n-2)$ steepness indices.

The new stability exponents represent an essential improvement with respect to Eq. (2); in particular, the dependence of a^{-1} on the number of degrees of freedom improves from quadratic to linear. It is also remarkable that, for $\alpha_1 = \dots = \alpha_n = 1$ (quasi-convex case), we obtain the “optimal” stability exponents proved in ([16], [15], [27]), without using the local inversion of the frequency map, nor the Hamiltonian as a Lyapunov function.

A precise and fully quantitative formulation is given in the following paragraph.

B. Statement of the result. A C^1 function $h(I)$ is said to be steep in $U \subseteq \mathbb{R}^n$ with steepness indices $\alpha_1, \dots, \alpha_{n-1} \geq 1$ and (strictly positive) steepness coefficients C_1, \dots, C_{n-1} and r , if $\inf_{I \in U} \|\omega(I)\| > 0$ and, for any $I \in U$, for any j -dimensional linear subspace $\Lambda \subseteq \mathbb{R}^n$ orthogonal to $\omega(I)$ with $1 \leq j \leq n-1$, one has⁵

$$\max_{0 \leq \eta \leq \xi} \min_{u \in \Lambda: \|u\| = \eta} \|\pi_\Lambda \omega(I + u)\| \geq C_j \xi^{\alpha_j} \quad \forall \xi \in (0, r], \quad (3)$$

where π_Λ denotes the orthogonal projection over Λ .

To deal properly with initial data near the boundary, we will use the following notation: for any $\eta > 0$ and any $D \subseteq \mathbb{R}^n$, we let $D - \eta := \{I \in D : \overline{B(I, \eta)} \subseteq D\}$, where

$$B(I, \eta) = \{I' \in \mathbb{R}^n : \|I' - I\| < \eta\}$$

³In the paper [23] there is a statement concerning improved values for the stability exponents, however, the proof appears to have a serious gap and such values are not justified; see [25].

⁴The cases $n \leq 2$ are, in general, totally stable and therefore are not included in our analysis.

⁵For any vector $u \in \mathbb{C}^n$ we denote by $\|u\| := \sqrt{\sum_i |u_i|^2}$ its hermitean norm and by $|u| = \sum_i |u_i|$.

is the real euclidean ball centered in I of radius η and $\overline{B(I, \eta)}$ its closure.

Theorem 1. *Let H in (1) be real-analytic with h steep in U with steepness indices $\alpha_1, \dots, \alpha_{n-1}$ and let*

$$p_1 := \prod_{k=1}^{n-2} \alpha_k, \quad a := \frac{1}{2np_1}, \quad b := \frac{a}{\alpha_{n-1}}.$$

Then, there exist positive constants $\varepsilon_0, R_0, T, c > 0$ such that for any $0 \leq \varepsilon < \varepsilon_0$ the solution (I_t, φ_t) of the Hamilton equations for $H(I, \varphi)$ with initial data (I_0, φ_0) with $I_0 \in U - 2R_0\varepsilon^b$ satisfies

$$\|I_t - I_0\| \leq R_0\varepsilon^b \quad (4)$$

for any time t satisfying:

$$|t| \leq \frac{T}{\sqrt{\varepsilon}} \exp\left(\frac{c}{\varepsilon^a}\right). \quad (5)$$

C. Quantitative formulation. Next, we provide explicit estimates for the parameters ε_0, R_0, T, c appearing in Theorem 1.

To do this, we need to introduce some notations. Given “extension parameters” $\eta, \sigma > 0$ and any set D , we let the “extended complex domains” be defined by:

$$D_\eta = \bigcup_{I' \in D} \{I \in \mathbb{C}^n : \|I - I'\| \leq \eta\} \quad \text{and} \quad \mathbb{T}_\sigma^n = \{\varphi \in \mathbb{C}^n / (2\pi\mathbb{Z})^n : |\operatorname{Im} \varphi_i| \leq \sigma\}.$$

For any real action-angle function $u(I, \varphi)$ analytic in $D_\eta \times \mathbb{T}_\sigma^n$, with Fourier harmonics $u_k(I)$, we denote its Fourier-norm

$$|u|_{\eta, \sigma} = \sum_{k \in \mathbb{Z}^n} |u_k|_{D_\eta} e^{|k|\sigma},$$

where $|\cdot|_{D_\eta}$ denotes the sup-norm in D_η ; if it needs to be specified, we shall also use the heavier notation $|u|_{D; \eta, \sigma}$.

Let H be real-analytic in $U \times \mathbb{T}^n$ with h steep in U with steepness indices $\alpha_1, \dots, \alpha_{n-1}$ and steepness coefficients C_1, \dots, C_{n-1} and r . Without loss of generality, we can take the extension parameter in action space to be equal to the steepness coefficient r and we can find positive constants $s, \underline{\omega}, \overline{\omega}$ and M such that:

- $h(I)$ is real analytic on an open set which contains U_r ;
- $f(I, \varphi)$ is real analytic on an open set which contains $U_r \times \mathbb{T}_s^n$;
- For any $I \in U$, we have:

$$\underline{\omega} \leq \|\omega(I)\| \leq \overline{\omega}$$

and, for any $I_1, I_2 \in U_r$, we have:

$$\|\omega(I_1) - \omega(I_2)\| \leq M\|I_1 - I_2\|.$$

Now, for $1 \leq j \leq n-2$, let

$$p_j := \prod_{k=j}^{n-2} \alpha_k, \quad q_j := np_j - j, \quad \beta_j := \alpha_j + j(\alpha_j - 1), \quad (6)$$

and define the parameters

$$\kappa_j := \frac{\underline{\omega}}{M} \left(\frac{C_j}{\underline{\omega}} \right)^{\frac{1}{\alpha_j}} + 4 \left(2 \frac{2\bar{\omega} + Mr}{\underline{\omega}} \right)^{\frac{1}{\alpha_j}}, \quad (7)$$

$$E := \max \left(\max_{j \leq n-2} \left(\frac{(4M\kappa_j)^{\alpha_j} 6^{q_j(\alpha_j-1)}}{C_j \left(\frac{\underline{\omega}}{2\sqrt{2}} \right)^{\alpha_j-1}} \right)^{\frac{1}{\beta_j}}, 4 \right). \quad (8)$$

Then, in Theorem 1, one can take

$$\begin{aligned} \varepsilon_* &:= \frac{1}{2^8} \frac{1}{6^{4np_1-5} E^{2np_1-1}} \frac{\underline{\omega}^2}{M|f|_{r,s}} \\ \varepsilon_0 &:= \varepsilon_* \min \left(\left(\frac{6\sqrt{2} Mr}{n \underline{\omega}} \right)^{\frac{1}{b}}, \left(\frac{18\sqrt{2}}{n} \right)^{\frac{1}{b}}, \left(\frac{r}{4n\kappa_{n-1}} \right)^{\frac{1}{b}} \left(\frac{12\sqrt{2} EC_{n-1}}{\underline{\omega}} \right)^{\frac{1}{a}}, \left(\frac{s}{6} \right)^{\frac{1}{a}}, 1 \right) \\ c &:= \varepsilon_*^a \frac{s}{6} \\ R_0 &:= \frac{r n \mu_0}{\varepsilon_*^b}, \text{ with } \mu_0 := \max \left(\frac{1}{24\sqrt{2}} \frac{\underline{\omega}}{Mr}, \frac{1}{6^2 2\sqrt{2}}, \frac{\kappa_{n-1}}{r} \left(\frac{\underline{\omega}}{12\sqrt{2} EC_{n-1}} \right)^{\frac{1}{\alpha_{n-1}}} \right) \\ T &:= \frac{s}{24\sqrt{2}} \frac{\underline{\omega}}{M(6E)^{\frac{1}{a}} \sqrt{\varepsilon_*} |f|_{r,s}}. \end{aligned} \quad (9)$$

D. On the proof. The proof of Nekhoroshev's theorem, in its various settings, can be split into:

a *geometric part*, devoted to the analysis of distribution of small divisors in action-space;
an *analytic part*, devoted to the construction of normal forms;
a *stability argument* yielding the confinement of the actions.

While the analytic part is obtained by adapting averaging methods to an analytic setting, the heart of Nekhoroshev's theorem resides in its geometric part. The geometric part of the steep case presented in [19, 20] still needed a deep revisitation, which is performed here and leads, in particular, to substantially improved stability exponents.

The proof of Theorem 1 will be obtained by deeply revisiting the geometric part of ([19, 20]). The essential improvement are the following.

First, we extend Pöschel's Geometric Lemma (see [27]) to allow for a more general power-law scaling of the amplitudes of the resonance domains. In this way, we allow for a definition of the resonance domains which depends on the euclidean volume (of a minimal cell) of the lattice generating the resonance, and is compatible with steepness indices $\alpha_i > 1$. In contrast with the convex case, the analog of Pöschel's Geometric Lemma is here far to accomplish the geometric part of the theorem. In fact, motions with initial conditions characterized by a given resonance, may move along preferential planes of the action space, called fast drift planes. In particular, one needs to extend in the action

space with fast drift planes the resonant domains obtained by a pull back from the frequency space: eventual degeneracies of the frequency map, which are typical of the steep non-convex case, may produce topologically complicate sets. Nevertheless, a regularity of the distribution of these extended resonant sets must be proved: this is needed in order to grant the non overlapping of resonant domains of the same multiplicity. In [19, 20], the non-overlapping is granted simply by construction of the resonance domains, but the price paid was an overestimate of resonant domains with the consequence of a strong n^{-2} scaling of the stability exponent (2). Here, we do not grant the overlapping by construction but, with a careful analysis of the topology of these sets, we obtain a better balance between optimal definition of resonant domains and their non-overlapping. Finally, our geometric construction is fully compatible with the usual analytic part and stability argument, such as the so called *resonance trap* of [19, 20]), and its improved version introduced in [5].

E. The paper is organized as follows. The main part (i.e., the geometric analysis) is presented in § 2: in § 2.1 we introduce several auxiliary parameters (needed to measure various covering sets, small divisors, cut-offs in Fourier space, time scales, etc.) and point out the relevant relations among them (relations, which, although based on simple calculus, are proven, for completeness in Appendix B). In § 2.2, merging and extending the geometric analysis of [19] and [27], we introduce a covering in action-space formed by (a suitable scale of) resonant and non-resonant regions. Section 2.3 is the heart of the paper, where the relevant analytic properties of the resonant and non-resonant regions are proven; the section is divided into three lemmata: the first is about geometric estimates concerning resonant domains; the second deals with small divisor estimates and the third one is a non-overlapping result for resonant regions corresponding to resonances of the same dimension. In § 3 we recall briefly Pöschel's normal form theory⁶ [27] and show how it can be used in our setting. In the final section 4 we put all pieces together and prove Theorem 1 with the constants listed in **C** above. In Appendix A we briefly review the notion of angles between linear spaces and, as mentioned above, Appendix B is an elementary check of the main relations among the auxiliary parameters.

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2 Geometry of resonances

2.1 Auxiliary parameters

In the proof of the Theorem 1 several auxiliary parameters will occur; in this section we define such parameters and point out some compatibility relations expressed as in inequalities, which will be needed in the following.

⁶Incidentally, real-analyticity is needed only here; in the geometric analysis C^2 -regularity is enough.

$$K := \left(\frac{\varepsilon_*}{\varepsilon}\right)^a \quad (10)$$

$$R := 2R_0\varepsilon^b \quad (11)$$

$$\rho := \frac{R}{2n}, \quad (12)$$

$$\widehat{\omega} := \frac{\omega}{2\sqrt{2}} \quad (13)$$

$$q_n := 0, \quad q_{n-1} := 1, \quad a_{n-1} := 1, \quad a_j := q_j - q_{j+1} \quad (1 \leq j \leq n-2). \quad (14)$$

Notice that the q_j 's are strictly decreasing since $a_j \geq 1$, indeed:

$$a_{n-2} = n(\alpha_{n-2} - 1) + 1; \quad a_j = np_{j+1}(\alpha_j - 1) + 1, \quad (1 \leq j \leq n-3). \quad (15)$$

Let Λ be any maximal K -lattice over \mathbb{Z}^n of dimension⁷ $1 \leq j \leq n-1$, $|\Lambda|$ its volume, and set:

$$\lambda_j := \frac{\widehat{\omega}}{(AK)^{q_j}}, \quad \text{where } A := 6E \quad (16)$$

$$r_j := \kappa_j \left(\frac{\lambda_j}{C_j}\right)^{\frac{1}{\alpha_j}} \quad (17)$$

$$\delta_\Lambda := \frac{\lambda_j}{|\Lambda|} \quad (18)$$

$$r_\Lambda := \frac{\delta_\Lambda}{M} \quad (19)$$

$$\gamma_\Lambda := (EK)^{a_j} \delta_\Lambda \quad (20)$$

$$R_\Lambda := \frac{\gamma_\Lambda}{4MK}, \quad (21)$$

Finally, we set

$$r_0 := \frac{\lambda_1}{2MK} \quad (22)$$

$$T_0 := \frac{sr_0}{5\varepsilon|f|_{r,s}} e^{K\frac{s}{6}}, \quad T_\Lambda := \frac{es}{24\varepsilon|f|_{r,s}} \frac{r_\Lambda}{e^{K\frac{s}{6}}}, \quad T_j := \min_{\Lambda: \dim \Lambda = j} T_\Lambda \quad (23)$$

$$T_{\text{exp}} := \min_{i=0, \dots, n-1} T_i. \quad (24)$$

It is then easy to check (see Appendix B) that under the assumption of Theorem 1, namely, $0 \leq \varepsilon < \varepsilon_0$, for any maximal K -lattice of dimension $1 \leq j \leq n-1$ (unless

⁷We recall that a “maximal K -lattice” Λ is a lattice which admits a basis of vectors $\tilde{k} \in \mathbb{Z}^n$ with $|\tilde{k}| := \sum_{i=1}^n |\tilde{k}_i| \leq K$, and it is not properly contained in any other lattice of the same dimension; the volume $|\Lambda|$ of the lattice Λ is defined as the euclidean volume of the parallelepiped spanned by a basis for Λ ; (see [27]). Notice that for any K -lattice of dimension j , one has $1 \leq |\Lambda| \leq K^j$.

otherwise specified) one has:

$$A \geq \max \left(\max_{j \in \{1, \dots, n-1\}} \left((E^{a_j} + 1)^2 + 1 \right)^{\frac{1}{2a_j}}, \left(\frac{4}{E^{a_j}} + 2 \right)^{\frac{1}{a_j}} \right) \quad (25)$$

$$Ks \geq 6 \quad (26)$$

$$r_\Lambda \leq \min \left(\frac{\rho}{2}, R_\Lambda \right), \quad (27)$$

$$\delta_\Lambda \leq \min \left(\frac{\omega}{r} \frac{\rho}{4}, \frac{\omega}{2r} (\rho - r_\Lambda), \hat{\omega} \right), \quad (j \leq n-2) \quad (28)$$

$$KM\kappa_j \left(\frac{\delta_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} \leq \frac{1}{4} \gamma_\Lambda, \quad (j \leq n-2) \quad (29)$$

$$R_\Lambda \leq r, \quad (30)$$

$$\varepsilon |f|_{r,s} \leq \min \left(\frac{1}{2^8} \frac{\lambda_1 r_0}{K}, \frac{\gamma_\Lambda r_\Lambda}{2^9 K}, \frac{\gamma_\Lambda R_\Lambda}{2^9 K} \right) \quad (31)$$

$$\max_{0 \leq i \leq n-1} r_i \leq \rho \quad (32)$$

$$r_0 \leq r \quad (33)$$

$$\sum_{j=0}^{n-1} r_j \leq \frac{R}{2} \quad (34)$$

$$R \leq \frac{r}{2}, \quad (35)$$

$$T_{\text{exp}} \geq \frac{T}{\sqrt{\varepsilon}} \exp \left(\frac{Ks}{6} \right). \quad (36)$$

2.2 Resonant and non-resonant domains

Fix $I_0 \in U - R$ and consider the set:

$$B := B(I_0, R) \subseteq U.$$

In order to prove the stability of all motions with initial actions I_0 , we need to cover the domain B with open domains where suitable normal forms adapted to the local resonance properties may be constructed. We here introduce resonant zones and resonant blocks as in [27], but, since we do not require any local inversion for the frequency map $\omega(I)$ (as it is typical of steepness [19, 20], see also [12]), these domains are directly defined in the action-space, without using any pull-back from a frequencies space. Then, we define suitable extensions, in the spirit of the original construction of [19] (see also [5]).

We first define the resonant zones and blocks depending on the parameter $K \geq 1$, representing a cut-off for the resonance order, and also on the parameters $0 < \lambda_1 < \dots < \lambda_{n-1} < \hat{\omega}$ defined above. As in [27], we consider only the resonances defined by

$$k \cdot \omega(I) = 0$$

with k in some maximal K -lattice $\Lambda \subseteq \mathbb{Z}^n$. We define the **resonant zone**

$$\mathcal{Z}_\Lambda := \{I \in B : \|\pi_{\langle \Lambda \rangle} \omega(I)\| < \delta_\Lambda\}, \quad (37)$$

where $\langle \Lambda \rangle$ denotes the real vector space spanned by the lattice Λ , and the **resonant block**

$$B_\Lambda := \mathcal{Z}_\Lambda \setminus \mathcal{Z}_{j+1} \quad , \quad j = \dim \Lambda, \quad (38)$$

where:

$$\mathcal{Z}_i := \cup_{\{\Lambda' : \dim \Lambda' = i\}} \mathcal{Z}_{\Lambda'}.$$

We also define $\mathcal{Z}_0 := B$ and the **non-resonant block** B_0 by

$$B_0 := \mathcal{Z}_0 \setminus \mathcal{Z}_1.$$

We remark that, since $\|\omega(I)\| \geq \underline{\omega} > \widehat{\omega} \geq \delta_\Lambda$ for any $I \in B$, the completely resonant zone $\mathcal{Z}_{\mathbb{Z}^n}$ is empty and so is \mathcal{Z}_n . This implies

$$B_\Lambda = \mathcal{Z}_\Lambda \quad , \quad \forall \Lambda \text{ s.t. } \dim \Lambda = n - 1. \quad (39)$$

Furthermore, if one defines

$$B_j := \cup_{\{\Lambda' : \dim \Lambda' = j\}} B_{\Lambda'} \quad ,$$

one sees immediately that

$$B_j = \mathcal{Z}_j \setminus \mathcal{Z}_{j+1},$$

so that, for any $1 \leq j \leq n - 1$, we have:

$$B = B_0 \cup B_1 \cup \dots \cup B_{j-1} \cup \mathcal{Z}_j, \quad (40)$$

and, in particular,

$$B = B_0 \cup B_1 \cup \dots \cup B_{n-1}. \quad (41)$$

Next, following Nekhoroshev, we introduce **discs**

$$\mathcal{D}_{\Lambda, \eta}^\rho(I) := \left(\left(\bigcup_{I' \in I + \langle \Lambda \rangle} B(I', \eta) \right) \cap \mathcal{Z}_\Lambda \cap (B - \rho) \right)^I \subseteq \mathcal{Z}_\Lambda \cap (B - \rho), \quad (42)$$

where $I + \langle \Lambda \rangle$ (called by Nekhoroshev, “fast drift plane”) denotes the plane through I parallel to $\langle \Lambda \rangle$, $(C)^I$ denotes the connected component of a set C which contains I and η is any positive number less or equal than ρ . The **extended resonant blocks** are then defined by⁸:

$$B_{\Lambda, r_\Lambda}^\rho := \bigcup_{I \in B_\Lambda \cap (B - \rho)} \mathcal{D}_{\Lambda, r_\Lambda}^\rho(I) \subseteq \mathcal{Z}_\Lambda \cap (B - \rho), \quad (43)$$

and the **extended non-resonant block** by:

$$B_0^\rho := B_0 \cap (B - \rho).$$

We remark that the set $B - \rho$ is not empty since $\rho < R$, and for any lattice Λ with $\dim \Lambda = n - 1$, we have, by (39), (43) and footnote 8,

$$B_{\Lambda, r_\Lambda}^\rho = B_\Lambda \cap (B - \rho) \quad , \quad (\dim \Lambda = n - 1). \quad (44)$$

⁸Notice that, if $I \in B_\Lambda$, then $I \in \mathcal{D}_{\Lambda, \eta}^\rho(I)$ so that $B_\Lambda \cap (B - \rho) \subseteq B_{\Lambda, r_\Lambda}^\rho$.

2.3 Geometric properties of the resonant domains

• Geometric estimates for resonant domains

For any maximal K -lattice Λ , we need to estimate the diameter of the intersection of the fast drift planes $I + \langle \Lambda \rangle$ with the resonant zones:

Lemma 2.1 *For any $I' \in B_\Lambda \cap (B - \rho)$ and $I'' \in \mathcal{C}_{\Lambda, r_\Lambda}^\rho(I')$ we have:*

$$\|I' - I''\| \leq \kappa_j \left(\frac{\delta_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} \leq r_j . \quad (45)$$

Proof We divide the proof of this lemma in three steps.

Step 1. Let $\tilde{\delta}, \tilde{\rho} > 0$ be such that

$$\tilde{\delta} \leq \min \left(\frac{\tilde{\rho}}{r}, \frac{1}{\sqrt{2}} \right) \omega , \quad (46)$$

and define

$$\mathcal{Z}_\Lambda(\tilde{\delta}) = \{I \in B : \|\pi_{\langle \Lambda \rangle} \omega(I)\| < \tilde{\delta}\} . \quad (47)$$

Let us also denote by $\langle \omega \rangle$ the linear space generated by $\omega(I)$; by $\langle \omega \rangle^\perp$ the linear space orthogonal to $\omega(I)$ and by $\Lambda_\omega = \pi_{\langle \omega \rangle^\perp} \langle \Lambda \rangle$ the linear space obtained by projecting every vector u of $\langle \Lambda \rangle$ on $\langle \omega \rangle^\perp$.

The first step will consist in proving that:

For any $I \in \mathcal{Z}_\Lambda(\tilde{\delta}) \cap (B - \tilde{\rho})$ and any $I' \in \left((I + \langle \Lambda \rangle) \cap \mathcal{Z}_\Lambda(\tilde{\delta}) \cap (B - \tilde{\rho}) \right)^I$ one has:

$$\|I - I'\| < 4 \left(\frac{2\overline{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}} . \quad (48)$$

Fix $I' \in \left((I + \langle \Lambda \rangle) \cap \mathcal{Z}_\Lambda(\tilde{\delta}) \cap (B - \tilde{\rho}) \right)^I$, with $I' \neq I$ (if $I' = I$ there is nothing to prove). Then, there exists a curve⁹ $u(t) \in \langle \Lambda \rangle$ such that $u(0) = 0$, $u(1) = I' - I$, and for any t , $I + u(t) \in \left((I + \langle \Lambda \rangle) \cap \mathcal{Z}_\Lambda(\tilde{\delta}) \cap (B - \tilde{\rho}) \right)^I$. In particular, $\|\pi_{\langle \Lambda \rangle} \omega(I + u(t))\| < \tilde{\delta}$.

The proof of (48) will be based on the following claims (i)÷(vii).

(i) Λ_ω is a vector space of dimension j .

Proof of (i): Clearly, if u_1, \dots, u_j is a basis for $\langle \Lambda \rangle$, then any vector in Λ_ω can be represented as a linear combination of $\pi_{\langle \omega \rangle^\perp} u_1, \dots, \pi_{\langle \omega \rangle^\perp} u_j \in \Lambda_\omega$. We prove that the vectors $\pi_{\langle \omega \rangle^\perp} u_1, \dots, \pi_{\langle \omega \rangle^\perp} u_j \in \Lambda_\omega$ are linearly independent, so that $\dim \Lambda_\omega = j$. First, we remark that the only vector u of $\langle \Lambda \rangle$ satisfying: $\pi_{\langle \omega \rangle^\perp} u = 0$ is $u = 0$. In fact, if there exists $u \neq 0$ such that $u \in \langle \Lambda \rangle$ and $\pi_{\langle \omega \rangle^\perp} u = 0$, then $\omega(I) \in \langle \Lambda \rangle$, and therefore we have:

$$\|\omega(I)\| = \|\pi_{\langle \Lambda \rangle} \omega(I)\| < \tilde{\delta} \leq \frac{\underline{\omega}}{\sqrt{2}},$$

⁹Notice that the set $\left((I + \langle \Lambda \rangle) \cap \mathcal{Z}_\Lambda(\tilde{\delta}) \cap (B - \tilde{\rho}) \right)^I$ is open in the relative topology of $I + \langle \Lambda \rangle$ and therefore is arc-connected in $I + \langle \Lambda \rangle$.

which is not possible since for any $I \in B$ we assumed $\|\omega(I)\| > \underline{\omega}$. Now, let us consider c_1, \dots, c_j such that: $\sum_{i=1}^j c_i \pi_{\langle \omega \rangle^\perp} u_i = 0$. Then, $\pi_{\langle \omega \rangle^\perp} \sum_i c_i u_i = 0$, and therefore $\sum_i c_i u_i = 0$. But, since the u_i are linearly independent, it follows $c_1, \dots, c_j = 0$.

(ii) For any $u \in \langle \Lambda \rangle$, we have $\pi_{\Lambda_\omega} u = \pi_{\langle \omega \rangle^\perp} u$.

Proof of (ii): We first compute:

$$\pi_{\langle \omega \rangle^\perp} u = \pi_{\Lambda_\omega} \pi_{\langle \omega \rangle^\perp} u + \pi_{\Lambda_\omega^\perp} \pi_{\langle \omega \rangle^\perp} u. \quad (49)$$

Since $\pi_{\langle \omega \rangle^\perp} u \in \Lambda_\omega$, we have $\pi_{\Lambda_\omega^\perp} \pi_{\langle \omega \rangle^\perp} u = 0$, so that (49) becomes:

$$\pi_{\langle \omega \rangle^\perp} u = \pi_{\Lambda_\omega} \pi_{\langle \omega \rangle^\perp} u. \quad (50)$$

But, $\pi_{\Lambda_\omega} u = \pi_{\Lambda_\omega} (\pi_{\langle \omega \rangle^\perp} u + \pi_{\langle \omega \rangle} u) = \pi_{\Lambda_\omega} \pi_{\langle \omega \rangle^\perp} u + \pi_{\Lambda_\omega} \pi_{\langle \omega \rangle} u$ and since $\Lambda_\omega \subseteq \langle \omega \rangle^\perp$, we have $\pi_{\Lambda_\omega} \pi_{\langle \omega \rangle} u = 0$, and therefore:

$$\pi_{\Lambda_\omega} u = \pi_{\Lambda_\omega} \pi_{\langle \omega \rangle^\perp} u. \quad (51)$$

From equations (50) and (51) we get (ii).

(iii) The angle¹⁰ between $\langle \Lambda \rangle$ and Λ_ω is equal to the angle between $\omega(I)$ and $\langle \Lambda \rangle^\perp$, in formulae:

$$\langle \Lambda \rangle \angle \Lambda_\omega = \omega(I) \angle \langle \Lambda \rangle^\perp. \quad (52)$$

Proof of (iii): By (ii) we have: $\langle \Lambda \rangle \angle \Lambda_\omega = \max_{u \in \langle \Lambda \rangle, u \neq 0} u \angle \pi_{\Lambda_\omega} u = \max_{u \in \langle \Lambda \rangle, u \neq 0} u \angle \pi_{\langle \omega \rangle^\perp} u = \langle \Lambda \rangle \angle \langle \omega \rangle^\perp$, and using (x) of Appendix A, we obtain $\langle \Lambda \rangle \angle \Lambda_\omega = \langle \Lambda \rangle \angle \langle \omega \rangle^\perp = \langle \omega \rangle \angle \langle \Lambda \rangle^\perp$.

(iv) For any t , one has $\|\pi_{\Lambda_\omega} \omega(I + u(t))\| < \frac{2\overline{\omega}\tilde{\delta}}{\underline{\omega}}$.

Proof of (iv): We start with

$$\begin{aligned} \|\pi_{\Lambda_\omega} \omega(I + u(t))\| &= \sqrt{\|\omega(I + u(t))\|^2 - \|\pi_{\Lambda_\omega^\perp} \omega(I + u(t))\|^2} \\ &= \|\omega(I + u(t))\| \sqrt{1 - |\cos(\omega(I + u(t)) \angle \Lambda_\omega^\perp)|^2} \\ &= \|\omega(I + u(t))\| |\sin(\omega(I + u(t)) \angle \Lambda_\omega^\perp)| \\ &\leq \overline{\omega} |\sin(\omega(I + u(t)) \angle \Lambda_\omega^\perp)| \end{aligned}$$

and then we produce an upper bound estimate of the angle $\omega(I + u(t)) \angle \Lambda_\omega^\perp$. By using property (ix) of Appendix A, we first obtain:

$$\omega(I + u(t)) \angle \Lambda_\omega^\perp \leq \omega(I + u(t)) \angle \langle \Lambda \rangle^\perp + \langle \Lambda \rangle^\perp \angle \Lambda_\omega^\perp. \quad (53)$$

Now, recalling that $\langle \Lambda \rangle$ and Λ_ω have the same dimension (claim (i) above), we see that by properties (x) and (xi) of Appendix A, $\langle \Lambda \rangle^\perp \angle \Lambda_\omega^\perp = \Lambda_\omega \angle \langle \Lambda \rangle = \langle \Lambda \rangle \angle \Lambda_\omega = \omega(I) \angle \langle \Lambda \rangle^\perp$. From (53), we therefore obtain:

$$\omega(I + u(t)) \angle \Lambda_\omega^\perp \leq \omega(I + u(t)) \angle \langle \Lambda \rangle^\perp + \omega(I) \angle \langle \Lambda \rangle^\perp. \quad (54)$$

Then, since:

$$|\sin(\omega(I + u(t)) \angle \langle \Lambda \rangle^\perp)| = \frac{\|\pi_{\langle \Lambda \rangle} \omega(I + u(t))\|}{\|\omega(I + u(t))\|} < \frac{\tilde{\delta}}{\underline{\omega}}$$

¹⁰The notion of angle between linear spaces is briefly reviewed in Appendix A.

$$|\sin(\omega(I) \angle \langle \Lambda \rangle^\perp)| = \frac{\|\pi_{\langle \Lambda \rangle} \omega(I)\|}{\|\omega(I)\|} < \frac{\tilde{\delta}}{\underline{\omega}}, \quad (55)$$

and $\tilde{\delta}/\underline{\omega} \leq 1/\sqrt{2}$, both angles are strictly smaller than $\pi/4$, their sum is strictly smaller than $\pi/2$, and since $\sin(x)$ is monotone in $[0, \pi/2]$, from (54) and standard trigonometry, we obtain:

$$\begin{aligned} |\sin \omega(I + u(t)) \angle \Lambda_\omega^\perp| &\leq |\sin(\omega(I + u(t)) \angle \langle \Lambda \rangle^\perp + \omega(I) \angle \langle \Lambda \rangle^\perp)| \\ &\leq |\sin(\omega(I + u(t)) \angle \langle \Lambda \rangle^\perp)| + |\sin(\omega(I) \angle \langle \Lambda \rangle^\perp)| < 2 \frac{\tilde{\delta}}{\underline{\omega}}. \end{aligned}$$

We therefore obtain: $\|\pi_{\Lambda_\omega} \omega(I + u(t))\| \leq \overline{\omega} |\sin \omega(I + u(t)) \angle \Lambda_\omega^\perp| < \frac{2\overline{\omega} \tilde{\delta}}{\underline{\omega}}$.

$$(v) \quad \|\pi_{\langle \omega \rangle} u(t)\| < \frac{\tilde{\delta}}{\underline{\omega}} \|u(t)\|.$$

Proof of (v): Since $u(t) \in \langle \Lambda \rangle$ and $I \in \mathcal{Z}_\Lambda(\tilde{\delta})$, we have:

$$\|\pi_{\langle \omega \rangle} u(t)\| = \frac{|\omega(I) \cdot u(t)|}{\|\omega(I)\|} = \frac{|\pi_{\langle \Lambda \rangle} \omega(I) \cdot u(t)|}{\|\omega(I)\|} < \frac{\tilde{\delta}}{\underline{\omega}} \|u(t)\|.$$

(vi) $I + \pi_{\langle \omega \rangle^\perp} u(t) \in B$.

Proof of (vi): Since $I, I + u(t) \in B - \tilde{\rho}$, we have $\|u(t)\| \leq 2r$ and, using (35), we obtain $\|u(t)\| \leq r$. Then, from (v) and (46), we have: $\|\pi_{\langle \omega \rangle} u(t)\| < \frac{\tilde{\delta}}{\underline{\omega}} \|u(t)\| \leq \frac{\tilde{\delta}}{\underline{\omega}} r \leq \tilde{\rho}$. Therefore, $I + \pi_{\langle \omega \rangle^\perp} u(t) \in B$.

(vii) $\xi := \|\pi_{\langle \omega \rangle^\perp} (I' - I)\| \in (0, r]$.

Proof of (vii): Let us first assume $\xi = 0$, that is $I' - I \in \langle \omega \rangle$ so that

$$I' - I = \omega(I) \frac{\|I' - I\|}{\|\omega(I)\|}.$$

Since $I' - I \in \langle \Lambda \rangle$ and $I' \neq I$, this would imply also $\omega(I) \in \langle \Lambda \rangle$, and therefore:

$$\|\omega(I)\| = \|\pi_{\langle \Lambda \rangle} \omega(I)\| < \tilde{\delta} \leq \frac{\underline{\omega}}{\sqrt{2}},$$

which is not possible since for any $I \in B$ we have $\|\omega(I)\| > \underline{\omega}$. Therefore we have $\xi > 0$. Then, we have

$$\xi = \|\pi_{\langle \omega \rangle^\perp} (I' - I)\| \leq \|I' - I\| = \|u(1)\| \leq r.$$

Now, we are ready to complete the proof of (48). Since $0 < \xi \leq r$, let $0 \leq \eta_* \leq \xi$ the η which realizes the maximum in the definition of the steepness index of dimension j , that is:

$$\min_{u \in \Lambda_\omega: \|u\|=\eta_*} \|\pi_{\Lambda_\omega} \omega(I + u)\| > C_j \xi^{\alpha_j}. \quad (56)$$

The curve $\pi_{\langle \omega \rangle^\perp} u(t) \in \Lambda_\omega$ joins I and $I + \pi_{\langle \omega \rangle^\perp} (I' - I)$, and therefore

$$[0, \xi] \subseteq \cup_{t \in [0, 1]} \|\pi_{\langle \omega \rangle^\perp} u(t)\|,$$

so that there exists $t_* \in [0, 1]$ such that $\|\pi_{\langle \omega \rangle^\perp} u(t_*)\| = \eta_*$. From (56) it follows:

$$\|\pi_{\Lambda_\omega} \omega(I + \pi_{\langle \omega \rangle^\perp} u(t_*))\| > C_j \xi^{\alpha_j}.$$

But using claims (iv) and (v) we also obtain:

$$\begin{aligned} \|\pi_{\Lambda_\omega} \omega(I + \pi_{\langle \omega \rangle^\perp} u(t_*))\| &\leq \|\pi_{\Lambda_\omega} \omega(I + u(t_*))\| + M \|\pi_{\langle \omega \rangle} u(t_*)\| < 2 \frac{\bar{\omega}}{\underline{\omega}} \tilde{\delta} + M \frac{\tilde{\delta}}{\underline{\omega}} \|u(t_*)\| \\ &< \frac{2\bar{\omega} + Mr}{\underline{\omega}} \tilde{\delta} \end{aligned}$$

so that

$$C_j \xi^{\alpha_j} < \frac{2\bar{\omega} + Mr}{\underline{\omega}} \tilde{\delta},$$

and therefore

$$\|\pi_{\langle \omega \rangle^\perp} (I' - I)\| = \xi < \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}}.$$

Using again (v), we obtain:

$$\begin{aligned} \|I' - I\| &\leq \|\pi_{\langle \omega \rangle^\perp} (I' - I)\| + \|\pi_{\langle \omega \rangle} (I' - I)\| \\ &< \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}} + \frac{\tilde{\delta}}{\underline{\omega}} \|I' - I\| \\ &\leq \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}} + \frac{1}{\sqrt{2}} \|I' - I\|, \end{aligned} \tag{57}$$

that is:

$$\|I' - I\| < \frac{1}{1 - \frac{1}{\sqrt{2}}} \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}} < 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}}.$$

This finishes the proof of (48).

Step 2. Next, we prove that:

For any $I \in \mathcal{Z}_\Lambda \cap (B - \rho)$ and any $I' \in \mathcal{D}_{\Lambda, r_\Lambda}^\rho(I)$, we have:

$$\|I - I'\| \leq r_\Lambda + 4 \left(\frac{2\bar{\omega} + Mr \delta_\Lambda + Mr_\Lambda}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}}. \tag{58}$$

Fix $I' \in \mathcal{D}_{\Lambda, r_\Lambda}^\rho(I)$. Since $\mathcal{D}_{\Lambda, r_\Lambda}^\rho(I)$ is open and connected, there exists a curve $I + u(t) \in \mathcal{D}_{\Lambda, r_\Lambda}^\rho(I)$, $t \in [0, 1]$, such that $I + u(0) = I$, $I + u(1) = I'$. Since $\mathcal{D}_{\Lambda, r_\Lambda}^\rho(I) \subseteq \mathcal{Z}_\Lambda$, we have: $\|\pi_{\langle \Lambda \rangle} \omega(I + u(t))\| < \delta_\Lambda$ for any $t \in [0, 1]$, and also $\|\pi_{\langle \Lambda \rangle^\perp} u(t)\| \leq r_\Lambda$. In fact, since $I + u(t) \in \mathcal{D}_{\Lambda, r_\Lambda}^\rho(I) \subseteq \cup_{\tilde{I} \in I + \langle \Lambda \rangle} B(\tilde{I}, r_\Lambda)$, there exists a curve $u'(t) \in \langle \Lambda \rangle$ such that $\|u(t) - u'(t)\| \leq r_\Lambda$, and therefore

$$\|\pi_{\langle \Lambda \rangle^\perp} u(t)\| = \|\pi_{\langle \Lambda \rangle^\perp} (u(t) - u'(t))\| \leq \|u(t) - u'(t)\| \leq r_\Lambda.$$

Then, we define $u''(t) := \pi_{\langle \Lambda \rangle} u(t)$, so that $u''(0) = \pi_{\langle \Lambda \rangle} u(0) = 0$, $I + u''(t) \in I + \langle \Lambda \rangle$, and

$$\|u''(t) - u(t)\| = \|\pi_{\langle \Lambda \rangle} u(t) - u(t)\| = \|\pi_{\langle \Lambda \rangle^\perp} u(t)\| \leq r_\Lambda.$$

Therefore, on the one hand we have $I + u''(t) \in B - \rho + r_\Lambda$, on the other hand:

$$\begin{aligned} \|\pi_{\langle \Lambda \rangle} \omega(I + u''(t))\| &\leq \|\pi_{\langle \Lambda \rangle} \omega(I + u(t))\| + \|\pi_{\langle \Lambda \rangle} (\omega(I + u''(t)) - \omega(I + u(t)))\| \\ &\leq \|\pi_{\langle \Lambda \rangle} \omega(I + u(t))\| + \|\omega(I + u''(t)) - \omega(I + u(t))\| \\ &\leq \|\pi_{\langle \Lambda \rangle} \omega(I + u(t))\| + M\|u''(t) - u(t)\| \leq \delta_\Lambda + Mr_\Lambda. \end{aligned}$$

Therefore, for any $t \in [0, 1]$, we have:

$$I + u''(t) \in \left((I + \langle \Lambda \rangle) \cap \mathcal{Z}_\Lambda(\delta_\Lambda + Mr_\Lambda) \cap (B - (\rho - r_\Lambda)) \right)^I.$$

We use, now, (48) (step 1) with

$$\tilde{\delta} := \delta_\Lambda + Mr_\Lambda \quad \text{and} \quad \tilde{\rho} := \rho - r_\Lambda.$$

In fact, $I \in \mathcal{Z}_\Lambda \subseteq \mathcal{Z}_\Lambda(\tilde{\delta})$; $I \in B - \rho \subseteq B - \tilde{\rho}$; from (28) it follows:

$$\tilde{\delta} \leq \min \left(\frac{\omega}{r} \tilde{\rho}, \frac{\omega}{\sqrt{2}} \right).$$

Therefore, we have:

$$\|u''(t)\| \leq 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\tilde{\delta}}{C_j} \right)^{\frac{1}{\alpha_j}} = 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda + Mr_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}},$$

for any $t \in [0, 1]$. In particular we have:

$$\|I' - I\| = \|u(1)\| \leq \|u''(1) - u(1)\| + \|u''(1)\| \leq r_\Lambda + 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda + Mr_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}}.$$

Step 3. We may conclude the proof of the lemma. From (27) and (28) we obtain

$$\delta_\Lambda + Mr_\Lambda = 2\delta_\Lambda \leq \frac{\omega\rho}{2r} \leq \frac{\omega}{r}(\rho - r_\Lambda),$$

so that applying (58) and using again (27), we have:

$$\|I' - I''\| \leq r_\Lambda + 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda + Mr_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} \leq \frac{\delta_\Lambda}{M} + 4 \left(2 \frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}}.$$

Then, since $\alpha_j \geq 1$ and (recall (28)) $\delta_\Lambda/\underline{\omega} < 1$, we have $(\delta_\Lambda/\underline{\omega}) \leq (\delta_\Lambda/\underline{\omega})^{\frac{1}{\alpha_j}}$, from which the first inequality in (45) follows at once; the second inequality follows from the fact that $\delta_\Lambda \leq \lambda_j$ and from the definition of r_j . \square

• Small divisor estimates

We recall ([27]) that a set $\tilde{B} \subseteq B$ is γ - K non resonant modulo Λ if we have $|k \cdot \omega(I)| \geq \gamma$ for any $k \in \mathbb{Z}^n \setminus \Lambda$ such that $|k| \leq K$; we will say that $\tilde{B} \subseteq B$ is γ -non resonant if $|k \cdot \omega(I)| \geq \gamma$ for any $k \in \mathbb{Z}^n \setminus \{0\}$ such that $|k| \leq K$. The following result is a generalization of the Geometric Lemma in [27].

Lemma 2.2 (i) For any maximal K -lattice Λ , the resonant block B_Λ is γ_Λ - K non resonant modulo Λ , while the non resonant block B_0 is λ_1 - K non resonant.

(ii) If $j = n - 1$, the extended block $B_{\Lambda, r_\Lambda}^\rho$ is γ_Λ - K non-resonant modulo Λ ; if $j \leq n - 2$, the extended block $B_{\Lambda, r_\Lambda}^\rho$ is $\gamma_\Lambda/2$ - K non-resonant modulo Λ .

Proof of (i): Let us first consider $I \in B_0$, so that $I \notin \mathcal{Z}_1$. For any $k \in \mathbb{Z}^n$, with $|k| \leq K$, let us denote by \tilde{k} the vector which generates the maximal one dimensional K -lattice containing k . Since $I \notin \mathcal{Z}_1$ we have:

$$\|\pi_{\langle \tilde{k} \rangle} \omega(I)\| \geq \frac{\lambda_1}{\|\tilde{k}\|} \geq \frac{\lambda_1}{\|k\|},$$

and consequently $|k \cdot \omega(I)| = \|k\| \|\pi_{\langle \tilde{k} \rangle} \omega(I)\| \geq \lambda_1$. Therefore, B_0 is λ_1 - K non resonant.

Now, consider a maximal K -lattice Λ , with $j := \dim \Lambda \in \{1, \dots, n - 1\}$ and let $I \in B_\Lambda$. As in [27], let $k \notin \Lambda$ with $|k| \leq K$ and denote by Λ_+ the maximal K -lattice generated by Λ and k (since Λ is maximal, $\dim \Lambda_+ = j + 1$). For the purpose of this proof, let us denote

$$\pi := \pi_{\langle \Lambda \rangle}, \quad \pi_\perp := \pi_{\langle \Lambda \rangle^\perp} = \text{Id} - \pi, \quad \pi_+ := \pi_{\langle \Lambda_+ \rangle},$$

where Id denotes the identity map. Since $\pi\pi_+ = \pi$, it is easy to check that

$$\pi_\perp k \cdot (\pi_+ \omega(I) - \pi \omega(I)) + \pi k \cdot \pi \omega(I) = k \cdot \omega(I).$$

Thus, since the vectors $\pi_\perp k$ and $\pi_+ \omega(I) - \pi \omega(I) = \pi_\perp \pi_+ \omega(I)$ are proportional, and $|\Lambda_+| \leq |\Lambda| \|\pi_\perp k\|$, we obtain

$$\begin{aligned} |k \cdot \omega(I)| &\geq |\pi_\perp k \cdot (\pi_+ \omega(I) - \pi \omega(I))| - |\pi k \cdot \pi \omega(I)| \\ &= \|\pi_\perp k\| \|\pi_+ \omega(I) - \pi \omega(I)\| - |\pi k \cdot \pi \omega(I)| \\ &\geq \frac{|\Lambda_+|}{|\Lambda|} \sqrt{\|\pi_+ \omega(I)\|^2 - \|\pi \omega(I)\|^2} - \|\pi k\| \|\pi \omega(I)\|. \end{aligned}$$

Using $\|\pi k\| \leq \|k\| \leq |k| \leq K$, $\|\pi \omega(I)\| < \delta_\Lambda$, $\|\pi \omega(I)\| \geq \delta_{\Lambda_+}$ we obtain:

$$|k \cdot \omega(I)| \geq \frac{|\Lambda_+|}{|\Lambda|} \sqrt{\frac{\lambda_{j+1}^2}{|\Lambda_+|^2} - \frac{\lambda_j^2}{|\Lambda|^2}} - K \delta_\Lambda.$$

Using again $|\Lambda_+| \leq |\Lambda|K$, and $K \leq K^{a_j}$, we obtain:

$$\begin{aligned} |k \cdot \omega(I)| &\geq \frac{1}{|\Lambda|} \left(\sqrt{\lambda_{j+1}^2 - K^2 \lambda_j^2} - K \lambda_j \right) \geq \delta_\Lambda \left(\sqrt{\left(\frac{\lambda_{j+1}}{\lambda_j} \right)^2 - K^2} - K \right) \\ &\geq \delta_\Lambda \left(\sqrt{(AK)^{2a_j} - K^2} - K \right) \geq \delta_\Lambda \left(\sqrt{(AK)^{2a_j} - K^{2a_j}} - K^{a_j} \right) \\ &= \delta_\Lambda K^{a_j} \left(\sqrt{A^{2a_j} - 1} - 1 \right), \end{aligned} \tag{59}$$

so that, by (25), we finally get:

$$|k \cdot \omega(I)| \geq \delta_\Lambda K^{a_j} \left(\sqrt{A^{2a_j} - 1} - 1 \right) \geq E^{a_j} K^{a_j} \delta_\Lambda = \gamma_\Lambda.$$

□

Proof of (ii): If $j = n - 1$, the conclusion follows directly from lemma 2.2-(i) and (44). Let us therefore consider, for any $j = 1, \dots, n - 2$, $I \in B_{\Lambda, r_\Lambda}^\rho$ and $I' \in B_\Lambda \cap (B - \rho)$ such that $I \in \mathcal{C}_{\Lambda, r_\Lambda}^\rho(I')$. By (28) and (58) we get

$$\|I' - I\| \leq r_\Lambda + 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda + Mr_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}}.$$

Using also lemma 2.2-(i), for any $k \in \mathbb{Z}^n \setminus \Lambda$ with $|k| \leq K$, we have

$$|k \cdot \omega(I)| \geq |k \cdot \omega(I')| - KM \|I - I'\| \geq \gamma_\Lambda - KM \left(r_\Lambda + 4 \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda + Mr_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} \right). \quad (60)$$

But, since, by (27) and (20), $KMr_\Lambda \leq \gamma_\Lambda/4$ from (29) there follows

$$\begin{aligned} 4KM \left(\frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda + Mr_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} &\leq 4KM \left(2 \frac{2\bar{\omega} + Mr}{\underline{\omega}} \frac{\delta_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} \\ &\leq KM \kappa_j \left(\frac{\delta_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} \leq \frac{\gamma_\Lambda}{4}, \end{aligned} \quad (61)$$

which, together with (60) yields $|k \cdot \omega(I)| \geq \gamma_\Lambda/2$. □

• Non overlapping of extended blocks and zones

Lemma 2.3 *For any maximal K -lattices $\Lambda \neq \Lambda'$ of the same dimension $j = 1, \dots, n - 1$, we have*

$$\overline{B_{\Lambda, r_\Lambda}^\rho} \cap \mathcal{Z}_{\Lambda'} = \emptyset.$$

Proof. Let $\Lambda \neq \Lambda'$ be maximal K -lattices of the same dimension $j \leq n - 1$ and consider $I \in \overline{B_{\Lambda, r_\Lambda}^\rho}$: we have to prove that $I \notin \mathcal{Z}_{\Lambda'}$, i.e.,

$$\|\pi_{\langle \Lambda' \rangle} \omega(I)\| \geq \delta_{\Lambda'}. \quad (62)$$

We divide the proof in two steps: the case $j \leq n - 2$ and the case $j = n - 1$.

Step 1. ($1 \leq j \leq n - 2$). The argument follows from the following claims (i)÷(vi).

(i) *For any $\eta > 0$, there exists $I' \in B_\Lambda \cap (B - \rho)$ such that $\|I - I'\| \leq \kappa_j \left(\frac{\delta_\Lambda}{C_j} \right)^{\frac{1}{\alpha_j}} + \eta$.*

Proof of (i): Since $I \in \overline{B_{\Lambda, r_\Lambda}^\rho}$, there exists $I'' \in B_{\Lambda, r_\Lambda}^\rho$ such that $\|I'' - I\| < \eta$; (by definition of $B_{\Lambda, r_\Lambda}^\rho$) there exists $I' \in B_\Lambda \cap (B - \rho)$ such that $I'' \in \mathcal{D}_{\Lambda, r_\Lambda}^\rho(I')$. Then, (i) immediately follows from (45).

(ii) $\|\pi_{\langle \Lambda' \rangle} \omega(I')\| \geq E^{a_j} K^{a_j-1} \delta_\Lambda$.

Proof of (ii): Since $\Lambda \neq \Lambda'$, there exists $k \in \Lambda'$ such that $k \notin \Lambda$ and $|k| \leq K$. Therefore we have $\|\pi_{\langle \Lambda' \rangle} \omega(I')\| \geq |k \cdot \omega(I')|/|k|$ and since $I' \in B_\Lambda$, (ii) follows from Lemma 2.2.

(iii) $\|\pi_{\langle \Lambda' \rangle} \omega(I)\| \geq \frac{1}{2} E^{a_j} K^{a_j-1} \delta_\Lambda$.

Proof of (iii): Choose $\eta \leq \frac{\gamma_\Lambda}{4KM}$. Then, by using (29), (i) and (ii), we obtain

$$\begin{aligned} \|\pi_{\langle \Lambda' \rangle} \omega(I)\| &\geq \|\pi_{\langle \Lambda' \rangle} \omega(I')\| - M\|I - I'\| \geq E^{a_j} K^{a_j-1} \delta_\Lambda - M\eta - M\kappa_j \left(\frac{\delta_\Lambda}{C_j}\right)^{\frac{1}{\alpha_j}} \\ &\geq E^{a_j} K^{a_j-1} \delta_\Lambda - \frac{\gamma_\Lambda}{2K} = \frac{1}{2} E^{a_j} K^{a_j-1} \delta_\Lambda. \end{aligned}$$

Now, observe that, if we have $\frac{1}{2} E^{a_j} K^{a_j-1} \delta_\Lambda \geq \delta_{\Lambda'}$, then (62) follows at once. Therefore, let us henceforth assume that

$$\frac{1}{2} E^{a_j} K^{a_j-1} \delta_\Lambda < \delta_{\Lambda'} \quad \text{i.e.} \quad \frac{|\Lambda'|}{|\Lambda|} < \frac{2}{E^{a_j} K^{a_j-1}}. \quad (63)$$

(iv) $\|\pi_{\langle \Lambda' \rangle} \omega(I')\| \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - 2\delta_\Lambda$.

Proof of (iv): Since $\Lambda \neq \Lambda'$, we consider $k \in \Lambda$ such that $k \notin \Lambda'$ and $|k| \leq K$, and we denote by Λ'' the maximal K -lattice of dimension $j+1$ which contains Λ' and k . For the purpose of the proof of (iv) let us denote:

$$\pi := \pi_{\langle \Lambda \rangle}, \quad \pi' := \pi_{\langle \Lambda' \rangle}, \quad \pi'' := \pi_{\langle \Lambda'' \rangle}.$$

First, since $I' \in B_\Lambda$, we have

$$\|\pi' \omega(I')\| \geq \|\pi'(\text{Id} - \pi)\omega(I')\| - \|\pi' \pi \omega(I')\| \geq \|\pi'(\text{Id} - \pi)\omega(I')\| - \delta_\Lambda. \quad (64)$$

Then, since $I' \in B_\Lambda$, $I' \notin \mathcal{Z}_{\Lambda''}$ and we have

$$\|\pi'' \omega(I')\| \geq \delta_{\Lambda''}. \quad (65)$$

Let us consider the vector $\nu = \pi_{\langle \Lambda' \rangle}^\perp k$. We remark that $\nu \in \langle \Lambda'' \rangle \setminus \{0\}$. In fact, on the one hand $k \notin \langle \Lambda' \rangle$, so that $\nu \neq 0$; on the other hand, since $\nu = k - \pi_{\langle \Lambda' \rangle} k$ is the sum of $k \in \langle \Lambda \rangle$ and of $-\pi' k \in \langle \Lambda' \rangle$, we have also $\nu \in \langle \Lambda'' \rangle$. Therefore, since ν is orthogonal to $\langle \Lambda' \rangle$, we have:

$$\pi_{\langle \Lambda'' \rangle} = \pi_{\langle \Lambda' \rangle} + \pi_{\langle \nu \rangle}. \quad (66)$$

Moreover, we have:

$$\frac{|\nu \cdot k|}{\|\nu\|} \geq \frac{|\Lambda''|}{|\Lambda'|}. \quad (67)$$

In fact, on the one hand we have

$$\frac{|\nu \cdot k|}{\|\nu\|} = \frac{|\pi_{\langle \Lambda' \rangle}^\perp k \cdot k|}{\|\pi_{\langle \Lambda' \rangle}^\perp k\|} = \|\pi_{\langle \Lambda' \rangle}^\perp k\|,$$

on the other hand we have

$$\|\pi_{\langle \Lambda' \rangle}^\perp k\| \geq \frac{|\Lambda''|}{|\Lambda'|}.$$

From (66), we obtain:

$$\begin{aligned} \|\pi'(\text{Id} - \pi)\omega(I')\|^2 &= \|\pi' \pi''(\text{Id} - \pi)\omega(I')\|^2 \\ &= \|\pi''(\text{Id} - \pi)\omega(I')\|^2 - \|\pi_{\langle \nu \rangle} \pi''(\text{Id} - \pi)\omega(I')\|^2 \end{aligned}$$

$$= \|\pi''(\text{Id} - \pi)\omega(I')\|^2 - \frac{|\nu \cdot \pi''(\text{Id} - \pi)\omega(I')|^2}{\|\nu\|^2}. \quad (68)$$

We notice that:

$$\pi''(\text{Id} - \pi)\omega(I') \neq 0. \quad (69)$$

In fact, first we have

$$\begin{aligned} \|\pi''(\text{Id} - \pi)\omega(I')\| &\geq \|\pi''\omega(I')\| - \|\pi''\pi\omega(I')\| \geq \delta_{\Lambda''} - \delta_{\Lambda} \\ &\geq \frac{\lambda_j}{|\Lambda''|} \left(A^{a_j} K^{a_j} - \frac{|\Lambda''|}{|\Lambda|} \right), \end{aligned}$$

then, using (67), (63), we obtain

$$\frac{|\Lambda''|}{|\Lambda|} = \frac{|\Lambda''|}{|\Lambda'|} \frac{|\Lambda'|}{|\Lambda|} \leq \|k\| \frac{|\Lambda'|}{|\Lambda|} < \frac{2\|k\|}{E^{a_j} K^{a_j-1}} \leq \frac{2K}{E^{a_j} K^{a_j-1}},$$

and therefore we have:

$$\|\pi''(\text{Id} - \pi)\omega(I')\| > \frac{\lambda_j}{|\Lambda''|} \left(A^{a_j} K^{a_j} - \frac{2K}{E^{a_j} K^{a_j-1}} \right).$$

Finally, since $K \geq 1$, $a_j \geq 1$, and using also (25), we have

$$\begin{aligned} \|\pi''(\text{Id} - \pi)\omega(I')\| &> \frac{\lambda_j}{|\Lambda''|} \left(A^{a_j} K - \frac{2K}{E^{a_j}} \right) = \frac{\lambda_j K}{|\Lambda''|} \left(A^{a_j} - \frac{2}{E^{a_j}} \right) \\ &\geq \frac{\lambda_j K}{|\Lambda''|} \left(2 + \frac{2}{E^{a_j}} \right) > 0. \end{aligned}$$

Therefore, from (68), (69), we have:

$$\|\pi'(\text{Id} - \pi)\omega(I')\| = \|\pi''(\text{Id} - \pi)\omega(I')\| \sqrt{1 - \frac{(\nu \cdot \pi''(\text{Id} - \pi)\omega(I'))^2}{\|\nu\|^2 \|\pi''(\text{Id} - \pi)\omega(I')\|^2}},$$

and, since

$$\pi''(\text{Id} - \pi)\omega(I') \cdot k = (\text{Id} - \pi)\omega(I') \cdot k = 0,$$

we obtain:

$$\|\pi'(\text{Id} - \pi)\omega(I')\| \geq \|\pi''(\text{Id} - \pi)\omega(I')\| \min_{u \in k^\perp, \|u\|=1} \sqrt{1 - \frac{(\nu \cdot u)^2}{\|\nu\|^2}}. \quad (70)$$

We remark that the maximum of $|\nu \cdot u| = |\pi_{\langle k^\perp \rangle} \nu \cdot u|$, for $u \in k^\perp$ and $\|u\| = 1$, is obtained for u parallel to $\pi_{\langle k^\perp \rangle} \nu$, that is for $u = \pi_{\langle k^\perp \rangle} \nu / \|\pi_{\langle k^\perp \rangle} \nu\|$. Therefore, we have:

$$\max_{u \in k^\perp, \|u\|=1} |\nu \cdot u| = \|\pi_{\langle k^\perp \rangle} \nu\|$$

and correspondingly:

$$\begin{aligned} \min_{u \in k^\perp, \|u\|=1} \sqrt{1 - \frac{(\nu \cdot u)^2}{\|\nu\|^2}} &= \sqrt{1 - \frac{\|\pi_{\langle k^\perp \rangle} \nu\|^2}{\|\nu\|^2}} = \sqrt{\frac{\|\nu\|^2 - \|\pi_{\langle k^\perp \rangle} \nu\|^2}{\|\nu\|^2}} = \frac{\|\pi_{\langle k \rangle} \nu\|}{\|\nu\|} \\ &= \frac{|\nu \cdot k|}{\|\nu\| \|k\|}. \end{aligned}$$

Therefore, from (70) and (67) we obtain

$$\|\pi'(\text{Id} - \pi)\omega(I')\| \geq \|\pi''(\text{Id} - \pi)\omega(I')\| \frac{|\nu \cdot k|}{\|\nu\| \|k\|} \geq \|\pi''(\text{Id} - \pi)\omega(I')\| \frac{|\Lambda''|}{|\Lambda'| \|k\|}.$$

Then, since: $\|\pi''(\text{Id} - \pi)\omega(I')\| \geq \|\pi''\omega(I')\| - \|\pi''\pi\omega(I')\| \geq \delta_{\Lambda''} - \delta_{\Lambda}$, we obtain:

$$\|\pi'(\text{Id} - \pi)\omega(I')\| \geq (\delta_{\Lambda''} - \delta_{\Lambda}) \frac{|\Lambda''|}{|\Lambda'| \|k\|} \geq \frac{\lambda_{j+1}}{|\Lambda'| \|k\|} - \delta_{\Lambda} \frac{|\Lambda''|}{|\Lambda'| \|k\|} \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - \delta_{\Lambda}, \quad (71)$$

and using (64) we obtain (iv).

We now are ready to finish the proof of (62) in the case $j \leq n - 2$. From inequalities (iv) and (i), we obtain:

$$\|\pi'\omega(I)\| \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - 2\delta_{\Lambda} - M \left(\kappa_j \left(\frac{\delta_{\Lambda}}{C_j} \right)^{\frac{1}{\alpha_j}} + \eta \right). \quad (72)$$

Using (29) and choosing $\eta \leq \frac{\gamma_{\Lambda}}{4KM}$, we obtain:

$$\|\pi'\omega(I)\| \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - 2\delta_{\Lambda} - \frac{1}{2} E^{a_j} K^{a_j-1} \delta_{\Lambda}. \quad (73)$$

Since we are assuming (63) and since $K^{a_j-1} \geq 1$, we obtain

$$\begin{aligned} \|\pi'\omega(I)\| &\geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - 2\delta_{\Lambda} - \frac{1}{2} E^{a_j} K^{a_j-1} \delta_{\Lambda} \\ &> A^{a_j} K^{a_j-1} \delta_{\Lambda'} - \frac{4}{E^{a_j} K^{a_j-1}} \delta_{\Lambda'} - \delta_{\Lambda'} > \left(A^{a_j} - \frac{4}{E^{a_j}} - 1 \right) \delta_{\Lambda'}, \end{aligned}$$

which, by (25), yields (62).

Step 2. We now consider maximal K -lattices $\Lambda \neq \Lambda'$ of the same dimension $j = n - 1$. Since $B_{\Lambda, r_{\Lambda}}^{\rho} = B_{\Lambda} \cap (B - \rho)$, we have $I \in \overline{B}_{\Lambda}$ and

$$\|\pi'\omega(I)\| \geq E^{a_j} K^{a_j-1} \delta_{\Lambda}. \quad (74)$$

In fact, since $\Lambda \neq \Lambda'$, there exists $k \in \Lambda'$ such that $k \notin \Lambda$ and $|k| \leq K$. Therefore we have $\|\pi'\omega(I)\| \geq |k \cdot \omega(I)| / \|k\|$ and since $I \in \overline{B}_{\Lambda}$, by lemma 2.2 we have

$$\|\pi'\omega(I)\| \geq \frac{|k \cdot \omega(I)|}{\|k\|} \geq E^{a_j} K^{a_j-1} \delta_{\Lambda}.$$

We also have:

$$\|\pi'\omega(I)\| \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - 2\delta_{\Lambda}. \quad (75)$$

First, since $I \in \overline{B}_{\Lambda}$, we have

$$\|\pi'\omega(I)\| \geq \|\pi'(\text{Id} - \pi)\omega(I)\| - \|\pi'\pi\omega(I)\| \geq \|\pi'(\text{Id} - \pi)\omega(I)\| - \delta_{\Lambda}. \quad (76)$$

Then, since $\Lambda \neq \Lambda'$, we consider $k \in \Lambda$ such that $k \notin \Lambda'$ and $|k| \leq K$. In particular, since $I \in \overline{B}_{\Lambda}$, we have

$$\|\omega(I)\| \geq \widehat{\omega}. \quad (77)$$

Let us consider the vector $\nu = \pi_{\langle \Lambda' \rangle^\perp} k$. Since ν is orthogonal to $\langle \Lambda' \rangle$, we have:

$$\text{Id} = \pi' + \pi_{\langle \nu \rangle}. \quad (78)$$

Moreover, we have:

$$\|\pi_{\langle \Lambda' \rangle^\perp} k\| = \frac{|\nu \cdot k|}{\|\nu\|} \geq \frac{1}{|\Lambda'|}. \quad (79)$$

In fact, since the K -lattice $\langle \Lambda', k \rangle$ is generated by Λ' and k is properly contained in \mathbb{Z}^n , we have

$$|\Lambda'| \|\pi_{\langle \Lambda' \rangle^\perp} k\| \geq |\langle \Lambda', k \rangle| \geq 1.$$

From (78), we obtain:

$$\|\pi(\text{Id} - \pi)\omega(I)\|^2 = \|(\text{Id} - \pi)\omega(I)\|^2 - \|\pi_{\langle \nu \rangle}(\text{Id} - \pi)\omega(I)\|^2,$$

and since:

$$\|\pi_{\langle \nu \rangle}(\text{Id} - \pi)\omega(I)\| = \frac{|\nu \cdot (\text{Id} - \pi)\omega(I)|}{\|\nu\|},$$

and:

$$\|(\text{Id} - \pi)\omega(I)\| \geq \widehat{\omega} - \delta_\Lambda \geq \widehat{\omega} - \lambda_{n-1} > 0,$$

we have:

$$\|\pi'(\text{Id} - \pi)\omega(I)\| = \|(\text{Id} - \pi)\omega(I)\| \sqrt{1 - \frac{(\nu \cdot (\text{Id} - \pi)\omega(I))^2}{\|\nu\|^2 \|(\text{Id} - \pi)\omega(I)\|^2}}.$$

Then, since $(\text{Id} - \pi)\omega(I) \cdot k = 0$, we have

$$\|\pi'(\text{Id} - \pi)\omega(I)\| \geq \|(\text{Id} - \pi)\omega(I)\| \min_{u \in k^\perp, \|u\|=1} \sqrt{1 - \frac{(\nu \cdot u)^2}{\|\nu\|^2}}. \quad (80)$$

We remark that the maximum of $|\nu \cdot u| = |\pi_{\langle k^\perp \rangle} \nu \cdot u|$, for $u \in k^\perp$ and $\|u\| = 1$, is obtained for u parallel to $\pi_{\langle k^\perp \rangle} \nu$, that is for $u = \pi_{\langle k^\perp \rangle} \nu / \|\pi_{\langle k^\perp \rangle} \nu\|$. Therefore, we have:

$$\max_{u \in k^\perp, \|u\|=1} |\nu \cdot u| = \|\pi_{\langle k^\perp \rangle} \nu\|$$

and correspondingly:

$$\begin{aligned} \min_{u \in k^\perp, \|u\|=1} \sqrt{1 - \frac{(\nu \cdot u)^2}{\|\nu\|^2}} &= \sqrt{1 - \frac{\|\pi_{\langle k^\perp \rangle} \nu\|^2}{\|\nu\|^2}} = \sqrt{\frac{\|\nu\|^2 - \|\pi_{\langle k^\perp \rangle} \nu\|^2}{\|\nu\|^2}} = \frac{\|\pi_{\langle k \rangle} \nu\|}{\|\nu\|} \\ &= \frac{|\nu \cdot k|}{\|\nu\| \|k\|}. \end{aligned}$$

Therefore, from (80), we obtain:

$$\|\pi'(\text{Id} - \pi)\omega(I)\| \geq \|(\text{Id} - \pi)\omega(I)\| \frac{|\nu \cdot k|}{\|\nu\| \|k\|},$$

and from (79) we obtain also:

$$\|\pi'(\text{Id} - \pi)\omega(I)\| \geq \|(\text{Id} - \pi)\omega(I)\| \frac{1}{|\Lambda'| \|k\|}.$$

Then, since: $\|(\text{Id} - \pi)\omega(I)\| \geq \|\omega(I)\| - \|\pi\omega(I)\| \geq \widehat{\omega} - \delta_\Lambda$, we obtain:

$$\|\pi'(\text{Id} - \pi)\omega(I)\| \geq (\widehat{\omega} - \delta_\Lambda) \frac{1}{|\Lambda'| \|k\|} \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - \delta_\Lambda, \quad (81)$$

and using (76) we obtain (75).

If $E^{a_j} K^{a_j-1} \delta_\Lambda \geq 2\delta_{\Lambda'}$, using (74), there is nothing more to prove. Therefore, we assume:

$$E^{a_j} K^{a_j-1} \delta_\Lambda < 2\delta_{\Lambda'}.$$

Then, using (75), we obtain:

$$\|\pi'\omega(I)\| \geq A^{a_j} K^{a_j-1} \delta_{\Lambda'} - \delta_\Lambda > A^{a_j} K^{a_j-1} \delta_{\Lambda'} - \frac{2}{E^{a_j} K^{a_j-1}} \delta_{\Lambda'}. \quad (82)$$

Since $K^{a_j-1} \geq 1$, we have:

$$\|\pi'\omega(I)\| > \left(A^{a_j} - \frac{2}{E^{a_j}}\right) \delta_{\Lambda'},$$

and using (25) we obtain (62). □

3 Normal forms and dynamics in resonant blocks

The geometric construction of § 2 together with normal form theory allows to have some control of the dynamics in the extended blocks. We shall use normal form theory in the version given by Pöschel in [27]; see, in particular, the “Normal Form Lemma” at p. 192 of [27] (which we shall use with parameters $p = q = 2$); notice that the constant M used in [27] is an upper bound on the derivative of $\omega(I)$, which is used only as Lipschitz constant, so that our notation is consistent with that used in [27].

In fact, the following lemma holds

Lemma 3.1 (i) *Let (I_t, φ_t) be the solution of the Hamilton equations with initial condition¹¹ $(\bar{I}_0, \varphi_0) \in B_0^\rho \times \mathbb{T}^n$. Then,*

$$\|I_t - \bar{I}_0\| \leq r_0 \quad (83)$$

for all times¹² $|t| \leq T_0$.

(ii) *Let Λ be a maximal K -lattice of dimension $j \in \{1, \dots, n-1\}$, and let (I_t, φ_t) be the solution of the Hamilton equations with initial data $(\bar{I}_0, \varphi_0) \in (B_\Lambda \cap (B - (j+1)\rho)) \times \mathbb{T}^n$. Let τ_e be the (possibly infinite) exit time from¹³ $B_{\Lambda, r_\Lambda}^\rho$. Then, if $|\tau_e| \geq T_\Lambda$, we have $\|I_t - \bar{I}_0\| \leq r_j$ for any time $|t| < T_\Lambda$; otherwise, there exists $0 \leq i \leq j-1$ such that $I_{\tau_e} \in B_i \cap (B - j\rho)$.*

¹¹I.e. $I_t|_{t=0} = \bar{I}_0$: we are using here a slight abuse of notation in order not to confuse the point I_0 in the statment of Theorem 1 with the arbitrary point \bar{I}_0 used here.

¹²Recall the definition of T_0 in (23).

¹³I.e., τ_e is such that $I_t \in B_{\Lambda, r_\Lambda}^\rho$ for $|t| < |\tau_e|$ and $I_{\tau_e} \notin B_{\Lambda, r_\Lambda}^\rho$.

Proof of (i): The non-resonant block B_0 is λ_1 non-resonant (see Lemma 2.2). Let us consider as extension vector (r_0, s) . Because of the definition of r_0 , (33), (26) and the first inequality in (31), we can apply the normal form lemma in [27] in B_0 . It then follows at once (83) for all times $|t| \leq T_0$ with T_0 as in (23). \square

Proof of (ii): Let us first assume that $|\tau_e| \geq T_\Lambda$ and consider the extension vector (r_Λ, s) . By Lemma 2.2–(ii), the domain $B_{\Lambda, r_\Lambda}^\rho$ is $\gamma_\Lambda/2 - K$ non-resonant modulo Λ . Thus, since $r_\Lambda \leq r$ (by (30)), and because of (26), the definition of R_Λ and the third inequality in (31), we can apply the Normal Form Lemma in [27] (with $p = q = 2$), in $B_{\Lambda, r_\Lambda}^\rho$. Thus, there exists a canonical transformation:

$$\begin{aligned} \phi : (B_{\Lambda, r_\Lambda}^\rho)_{\frac{r_\Lambda}{2}} \times \mathbb{T}_{\frac{s}{6}}^n &\rightarrow (B_{\Lambda, r_\Lambda}^\rho)_{r_\Lambda} \times \mathbb{T}_s^n \\ (I', \varphi') &\mapsto (I, \varphi) = \phi(I', \varphi') \end{aligned} \quad (84)$$

conjugating H to its resonant normal form:

$$H_\Lambda = H \circ \phi = h + \varepsilon g + \varepsilon f_* \quad (85)$$

with g a real-analytic functions having the Fourier expansion

$$g = \sum_{k \in \Lambda} g_k \exp(ik \cdot \varphi), \quad (86)$$

and the “remainder” f_* satisfying the exponential bound:

$$|f_*|_{B_{\Lambda, r_\Lambda}^\rho; \frac{r_\Lambda}{2}, \frac{s}{6}} \leq e^{-K \frac{s}{6}} |f|_{r, s}. \quad (87)$$

Also, for any $(I', \varphi') \in (B_{\Lambda, r_\Lambda}^\rho)_{\frac{r_\Lambda}{2}} \times \mathbb{T}^n$, by the third inequality in (31), one has:

$$\|I' - I\| \leq \frac{8K}{\gamma_\Lambda} \varepsilon |f|_{r, s} \leq \frac{1}{2^6} r_\Lambda,$$

so that $\phi^{-1}(B_{\Lambda, r_\Lambda}^\rho \times \mathbb{T}^n) \subseteq (B_{\Lambda, r_\Lambda}^\rho)_{\frac{r_\Lambda}{2^6}} \times \mathbb{T}^n$. Finally, using the second inequality in (31) we have also:

$$\|I' - I\| \leq \frac{8K}{\gamma_\Lambda} \varepsilon |f|_{r, s} \leq \frac{1}{2^6} r_\Lambda.$$

Therefore, since $I_t \in B_{\Lambda, r_\Lambda}^\rho$ for any $|t| < |\tau_e|$, we may define $(I'_t, \varphi'_t) = \phi^{-1}(I_t, \varphi_t)$, and using the specific form of hamiltonian (85), we have

$$\|\pi_{\langle \Lambda^\perp \rangle}(I'_t - I'_0)\| \leq \varepsilon \left\| \int_0^t \frac{\partial f_*}{\partial \varphi}(I'_t, \varphi'_t) dt \right\| \leq \varepsilon |t| \sup_{(B_{\Lambda, r_\Lambda}^\rho)_{\frac{r_\Lambda}{2^6}} \times \mathbb{T}^n} \left\| \frac{\partial f_*}{\partial \varphi} \right\|.$$

By Cauchy estimate (see Lemma B.3 of [27]) and by (87), we have:

$$\sup_{(B_{\Lambda, r_\Lambda}^\rho)_{\frac{r_\Lambda}{2^6}} \times \mathbb{T}^n} \left\| \frac{\partial f_*}{\partial \varphi} \right\| \leq \frac{6}{\varepsilon s} |f_*|_{B_{\Lambda, r_\Lambda}^\rho; \frac{r_\Lambda}{2}, \frac{s}{6}} \leq \frac{6}{\varepsilon s} e^{-K \frac{s}{6}} |f|_{r, s},$$

so that, for any $|t| < T_\Lambda$, we have:

$$\|\pi_{\langle \Lambda^\perp \rangle}(I'_t - I'_0)\| \leq \frac{6\varepsilon}{es}|t|e^{-K\frac{s}{6}}|f|_{r,s} \leq \frac{1}{4}r_\Lambda.$$

As a consequence, the motion I_t has the representation:

$$I_t = \bar{I}_0 + v(t) + d(t) \quad (88)$$

with $v(t) \in \langle \Lambda \rangle$ with $v(0) = 0$, and $\|d(t)\| < \frac{3}{4}r_\Lambda$: indeed, we can write

$$I_t = \bar{I}_0 + (I_t - I'_t) + (I'_t - I'_0) + (I'_0 - \bar{I}_0),$$

and take $v(t) = \pi_{\langle \Lambda \rangle}(I'_t - I'_0)$ and $d(t) = (I_t - I'_t) + \pi_{\langle \Lambda \rangle^\perp}(I'_t - I'_0) + (I'_0 - \bar{I}_0)$. Therefore, $I_t \in B_{\Lambda, r_\Lambda}^\rho \subseteq \mathcal{Z}_\Lambda \cap (B - \rho)$ and because of the representation (88) the distance between I_t and the space $\bar{I}_0 + \langle \Lambda \rangle$ is smaller than $\frac{3}{4}r_\Lambda$. Furthermore, I_t is connected to \bar{I}_0 in the set $\left(\cup_{I' \in \bar{I}_0 + \langle \Lambda \rangle} B(I', \frac{3}{4}r_\Lambda) \right) \cap \mathcal{Z}_\Lambda \cap (B - \rho)$ so that $I_t \in \mathcal{C}_{\Lambda, \frac{3}{4}r_\Lambda}^\rho(\bar{I}_0) \subseteq \mathcal{C}_{\Lambda, r_\Lambda}^\rho(\bar{I}_0)$. Thus, by Lemma 2.1 we have $\|I_t - \bar{I}_0\| \leq r_j$ for any $|t| < T_\Lambda$, as claimed.

Let us now assume that the exit time τ_e satisfies: $0 < |\tau_e| < T_\Lambda$. Since for any time $|t| < |\tau_e|$, we have $I_t \in \mathcal{C}_{\Lambda, \frac{3}{4}r_\Lambda}^\rho(\bar{I}_0)$, we have also: $I_{\tau_e} \in \overline{\mathcal{C}_{\Lambda, \frac{3}{4}r_\Lambda}^\rho(\bar{I}_0)}$. As a consequence (again Lemma 2.1), we have $\|I_t - \bar{I}_0\| \leq r_j < \rho$ for any $|t| \leq |\tau_e|$ and since $\bar{I}_0 \in B - (j+1)\rho$, we also have

$$I_{\tau_e} \in B - j\rho. \quad (89)$$

Since $I_t \in \mathcal{C}_{\Lambda, \frac{3}{4}r_\Lambda}^\rho(\bar{I}_0)$, the distance between I_t and $\bar{I}_0 + \langle \Lambda \rangle$ is strictly smaller than $\frac{3}{4}r_\Lambda$, and the distance between I_{τ_e} and $\bar{I}_0 + \langle \Lambda \rangle$ is smaller or equal than $\frac{3}{4}r_\Lambda$. Finally, since $I_t \in \mathcal{C}_{\Lambda, \frac{3}{4}r_\Lambda}^\rho(\bar{I}_0)$, we have $I_t \in \mathcal{Z}_\Lambda$, that is: $\|\pi_{\langle \Lambda \rangle}\omega(I_t)\| < \delta_\Lambda$. As a consequence, since $I_{\tau_e} \notin B_{\Lambda, r_\Lambda}^\rho$, the only possibility is: $\|\pi_{\langle \Lambda \rangle}\omega(I_{\tau_e})\| = \delta_\Lambda$. But this means that $I_{\tau_e} \notin \mathcal{Z}_\Lambda$. On the other hand, by Lemma 2.3, I_{τ_e} cannot belong to any $\mathcal{Z}_{\Lambda'}$ for any maximal K -lattice $\Lambda' \neq \Lambda$ of the same dimension j ; therefore $I_{\tau_e} \notin \mathcal{Z}_j$, whence, by (40), there must exist an $i \in 0, \dots, j-1$ such that $I_{\tau_e} \in B_i$, which, together with (89), concludes the proof of the lemma. \square

4 The resonance trap argument and conclusion of the proof

We are now in position to conclude the proof of the theorem, proving (4) and (5).

In view of (41), there are two alternatives¹⁴:

- (a) either $I_0 \in B_0 \cap (B - n\rho)$;
- (b) or $I_0 \in B_\Lambda \cap (B - (j+1)\rho)$ for some maximal K -lattice of dimension $j \in \{1, \dots, n-1\}$.

In case (a), by Lemma 3.1–(i), by (32), the definition of T_0 and T_{\exp} ((23), (24)) and (36), the theorem is proved.

In case (b), by Lemma 3.1–(ii), there are two alternatives:

¹⁴Recall that I_0 is the center of $B \subseteq U$, which is a sphere of radius $r = 2R_0\varepsilon^b = 2n\rho$.

- (b1) either $\|I_t - I_0\| \stackrel{(32)}{\leq} r_j \leq \rho$, for $|t| \leq T_{\text{exp}} \leq T_\Lambda$
- (b2) or there exist a t_1 such that $\|I_t - I_0\| \leq \rho$ for all $|t| \leq |t_1|$ and $I_{t_1} \in B_i \cap (B - j\rho)$ for some $i \in \{0, \dots, j-1\}$.

In case (b1), by (23), (24) and (36), and recalling that $\rho = R_0 \varepsilon^b / n$, the theorem is proved. In case (b2) we iterate the above scheme. Hence, after $0 \leq k \leq n-1$ steps we see that the action-trajectory I_t ends up either in a “trapping resonant region” $B_\Lambda \subseteq B_i$ where it gets stuck for exponentially long times or it will end up in B_0 where also gets stucked for exponentially long time. Since in such k steps I_t moves at most by $k\rho$ we see that in the (possible) fast drift we have $\|I_t - I_0\| \leq k\rho \leq (n-1)\rho$ to which we have to add the displacement in the trapping region which is again at most ρ . Thus, for times $|t| \leq T_{\text{exp}}$ we have $\|I_t - I_0\| \leq n\rho = R_0 \varepsilon^b$ as claimed. \square

We remark that, in the case (b) above, I_t may visit several blocks in the time T_{exp} ; let us denote by j^* their minimal multiplicity, and $t^* < T_{\text{exp}}$ be such that $I_{t^*} \in B_{\Lambda^*} \cap (B - (j^* + 1)\rho)$ with $\dim \Lambda^* = j^*$. Then, we have $I_0 \in B_{\Lambda^*}$ (and, therefore, $I_t \in B_{\Lambda^*}$ for all $|t| \leq T_{\text{exp}}$). In fact, since the geometry of resonances of the Hamilton function $-H$ is identical to the geometry of resonances of H , if we consider the solution (I'_t, φ'_t) of the Hamilton equations of $-H$ with $I'_0 = I_{t^*}$, and apply Lemma (3.1), we obtain that $I_0 = I'_{-t^*} \in B_\Lambda \cap B_{\Lambda^*}$.

A Angles between linear spaces

In this appendix $n \geq 2$, u, v, w, z, \dots denote vectors in \mathbb{R}^n and L_1, L_2, L', \dots linear vector subspaces of \mathbb{R}^n of dimension $m \in \{1, \dots, n-1\}$; π_L denotese the orthogonal projection onto the linear space L and $\text{Arccos} : [-1, 1] \rightarrow [0, \pi]$ denotes the principal branch of the inverse real cosine.

Definition A.1 Let . The angle between u and v is defined as

$$u \angle v = \begin{cases} \text{Arccos} \frac{u \cdot v}{\|u\| \|v\|} , & \text{if } u, v \neq 0 \\ \frac{\pi}{2} . & \text{otherwise} . \end{cases}$$

Definition A.2 The angle between L_1 and L_2 is defined as

$$L_1 \angle L_2 := \max_{u \in L_1 \setminus \{0\}} u \angle \pi_{L_2} u .$$

We, next, list a few elementary properties of angles between linear spaces, whose simple proof is left to the reader (for the proof of items x and xi , see, also, [Nekhoroshev79, p. 45]).

- i. $u \angle v \in [0, \pi]$ and $u \angle v \in [0, \pi/2]$ if and only if $u \cdot v \geq 0$; $L_1 \angle L_2 \in [0, \pi/2]$.
- ii. $L_1 \angle L_2 = \frac{\pi}{2}$ if and only if¹⁵ $L_1 \cap L_2^\perp \neq \{0\}$.
 $L_1 \angle L_2 < \frac{\pi}{2}$ if and only if $\{u \in L_1 : \pi_{L_2} u = 0\} = \{0\}$.
- iii. $u \angle \pi_L u = \text{Arccos} \frac{\|\pi_L u\|}{\|u\|} , \quad \forall u \neq 0$.

¹⁵ $L^\perp := \{u \in \mathbb{R}^n : u \cdot v = 0, \forall v \in L\}$.

- iv. $u \angle \pi_L u + u \angle \pi_{L^\perp} u = \frac{\pi}{2}$, $\forall u \neq 0$.
- v. $u \angle \pi_L u = \min_{v \in L \setminus \{0\}} u \angle v$.
- vi. $\cos L_1 \angle L_2 = \min_{u \in L_1 \setminus \{0\}} \max_{v \in L_2 \setminus \{0\}} \frac{u \cdot v}{\|u\| \|v\|}$.
- vii. For any u and v one has¹⁶ $u \angle v = v \angle u$.
- viii. If $u \neq 0 \neq v$, $u \angle v$ coincides with the (Euclidean) length of the shortest geodesic (equivalently, shortest curve) on the unit sphere $S^{n-1} := \{\xi \in \mathbb{R}^n : \|\xi\| = 1\}$ having as end-points the projections of u and v on S^{n-1} .
- ix. $u \angle v \leq u \angle w + w \angle v$. Also: $L_1 \angle L_2 \leq L_1 \angle L_3 + L_3 \angle L_2$.
- x. $L_1 \angle L_2 = L_2^\perp \angle L_1^\perp$.
- xi. If $\dim L_1 = \dim L_2$, then $L_1 \angle L_2 = L_2 \angle L_1$.

B Parameter Relations

For completeness, in this appendix, we prove the elementary inequalities (25)÷(36). Recall the definitions of the parameters given in (7)÷(24).

First, we observe that from these definitions and the hypothesis $0 \leq \varepsilon \leq \varepsilon_0$, it follows easily:

$$E \geq 4 ; \quad A := 6E \geq 24 , \quad K := \left(\frac{\varepsilon_*}{\varepsilon} \right)^a \geq \left(\frac{\varepsilon_*}{\varepsilon_0} \right)^a \geq 1 ; \quad 1 \leq |\Lambda| \leq K^j ; \quad (\text{B.1})$$

$$\delta_\Lambda \leq \lambda_j ; \quad \rho = \frac{r\mu_0}{K^{1/\alpha_{n-1}}} ; \quad r = 2n\rho ; \quad (\text{B.2})$$

$$q_1 + 1 = \frac{1}{2a} \geq n ; \quad q_j \geq 2 \quad (j \leq n-2) ; \quad q_j \left(1 - \frac{1}{\alpha_j} \right) = a_j - 1 - j \left(1 - \frac{1}{\alpha_j} \right). \quad (\text{B.3})$$

(25): It follows immediately from (B.1).

(26): It follows from $\frac{\varepsilon_0}{\varepsilon_*} \leq \left(\frac{s}{6} \right)^{\frac{1}{a}}$.

(27): To get the 1st inequality observe that $q_j \geq 1 \geq 1/\alpha_{n-1}$, $r\mu_0 \geq \underline{\omega}/(24\sqrt{2}M)$ so that

$$\begin{aligned} r_\Lambda &= \frac{\underline{\omega}}{2\sqrt{2}} \frac{1}{(AK)^{q_j}} \frac{1}{|\Lambda|} \frac{1}{M} \leq \frac{\underline{\omega}}{2\sqrt{2}} \frac{1}{A} \frac{1}{K^{\frac{1}{\alpha_{n-1}}}} \frac{1}{M} \stackrel{(\text{B.1})}{\leq} \frac{\underline{\omega}}{48\sqrt{2}} \frac{1}{K^{\frac{1}{\alpha_{n-1}}}} \frac{1}{M} = \frac{1}{2} \frac{\underline{\omega}}{24\sqrt{2}M} \left(\frac{\varepsilon}{\varepsilon_*} \right)^b \\ &\leq \frac{r\mu_0}{2} \left(\frac{\varepsilon}{\varepsilon_*} \right)^b = \frac{\rho}{2} . \end{aligned}$$

As for the 2nd inequality we have: $r_\Lambda \leq r_\Lambda \frac{(EK)^{a_j}}{4K} = R_\Lambda$.

(28), *first inequality*: Using: $q_j \geq 2 \geq 1/\alpha_{n-1}$, (B.1) and $\mu_0 \geq 1/(6^2 2\sqrt{2}) > 2/(\sqrt{2}(24)^2)$, one finds

$$\delta_\Lambda = \frac{\underline{\omega}}{2\sqrt{2}} \frac{1}{(AK)^{q_j}} \frac{1}{|\Lambda|} \leq \frac{\underline{\omega}}{2\sqrt{2}} \frac{1}{24^2} \frac{1}{K^{\frac{1}{\alpha_{n-1}}}} \leq \frac{\underline{\omega}}{4} \frac{\mu_0}{K^{\frac{1}{\alpha_{n-1}}}} = \frac{\underline{\omega}\rho}{r4} .$$

(28), *second inequality*: $\delta_\Lambda \stackrel{(28)}{\leq} \frac{\underline{\omega}}{r} \frac{\rho}{4} \stackrel{(27)}{\leq} \frac{\underline{\omega}}{2r} (\rho - r_\Lambda)$.

¹⁶But, in general, $L_1 \angle L_2 \neq L_2 \angle L_1$: for example, if $n = 3$, $L_1 = \{(0, t, t) : t \in \mathbb{R}\}$ and $L_2 = \{x_3 = 0\}$, then $L_1 \angle L_2 = \pi/4$, while $L_2 \angle L_1 = \pi/2$.

(28), *third inequality*: $\delta_\Lambda = \frac{\underline{\omega}}{2\sqrt{2}} \frac{1}{(AK)^{q_j}} \leq \widehat{\omega}$.

(29): Using: the definitions given, the inequality $|\Lambda| \leq K^j$ and last equality in (B.3), one has:

$$\begin{aligned} \frac{KM\kappa_j \left(\frac{\delta_\Lambda}{C_j}\right)^{\frac{1}{\alpha_j}}}{\frac{1}{4}\gamma_\Lambda} &= \left(\frac{1}{C_j}\right)^{\frac{1}{\alpha_j}} (4M\kappa_j) K^{1-a_j+q_j\left(1-\frac{1}{\alpha_j}\right)} |\Lambda|^{1-\frac{1}{\alpha_j}} \left(\frac{2\sqrt{2}}{\underline{\omega}} A^{q_j}\right)^{1-\frac{1}{\alpha_j}} \frac{1}{E^{a_j}} \\ &\leq \left(\frac{1}{C_j}\right)^{\frac{1}{\alpha_j}} (4M\kappa_j) K^{q_j\left(1-\frac{1}{\alpha_j}\right)+1-a_j+\left(1-\frac{1}{\alpha_j}\right)} \left(\frac{2\sqrt{2}}{\underline{\omega}} A^{q_j}\right)^{1-\frac{1}{\alpha_j}} \frac{1}{E^{a_j}} \\ &= \left(\frac{(4M\kappa_j)^{\alpha_j} 6^{np_j-j}(\alpha_j-1)}{C_j \left(\frac{\underline{\omega}}{2\sqrt{2}}\right)^{\alpha_j-1}} \frac{1}{E^{\alpha_j+j(\alpha_j-1)}}\right)^{\frac{1}{\alpha_j}} \leq 1, \end{aligned}$$

where last inequality comes from the definition of E .

(30): Using: $q_j \geq 1$, $q_j - a_j \geq 0$, $E \geq 1$, $a/b = \alpha_{n-1} \geq 1$, one has:

$$\begin{aligned} \frac{R_\Lambda}{r} &= \frac{1}{8\sqrt{2}} \frac{1}{6^{q_j}} \frac{\underline{\omega}}{Mr} \frac{1}{K^{q_j-a_j+1}} \frac{1}{E^{q_j-a_j}} \frac{1}{|\Lambda|} \leq \frac{1}{48\sqrt{2}} \frac{\underline{\omega}}{Mr} \frac{1}{K} \stackrel{(B.1)}{\leq} \frac{1}{48\sqrt{2}} \frac{\underline{\omega}}{Mr} \left(\frac{\varepsilon_0}{\varepsilon_*}\right)^a \\ &\stackrel{(9)}{\leq} \frac{1}{48\sqrt{2}} \frac{\underline{\omega}}{Mr} \left(\min\left(1, \frac{6\sqrt{2}}{n} \frac{Mr}{\underline{\omega}}\right)\right)^{\frac{a}{b}} \leq \frac{1}{8n} < 1. \end{aligned}$$

(31), *first inequality*: Using: the definitions given, $q_1 + 1 = \frac{1}{2a} = np_1$, $\varepsilon K^{1/a} = \varepsilon_*$, the definition of ε_* and E , one has:

$$\frac{2^8 K}{\lambda_1 r_0} \varepsilon |f|_{r,s} = \frac{2^4}{6^{2np_1-3}} \frac{1}{E} \leq \frac{2^4}{6^3} \frac{1}{4} = \frac{1}{54} < 1,$$

where in the inequality we used $n \geq 3$, $p_1 \geq 1$ and $E \geq 4$.

(31), *second inequality*: By the first inequality in (31), we see that the second inequality is implied by

$$\frac{1}{2^8} \frac{\lambda_1 r_0}{K} \leq \frac{\gamma_\Lambda r_\Lambda}{2^9 K}.$$

Now, using: the definitions given, $|\Lambda| \leq K^j$, $q_1 - q_j = n(p_1 - p_j) + (j - 1) \geq 0$ and the relation $a_j + 1 + 2(q_1 - q_j - j) = a_j - 1 + 2n(p_1 - p_j) \geq 0$, one has:

$$\frac{1}{2^8} \frac{\lambda_1 r_0}{K} \frac{2^9 K}{\gamma_\Lambda r_\Lambda} = \frac{1}{E^{a_j}} \frac{1}{A^{2(q_1-q_j)}} \frac{|\Lambda|^2}{K^{a_j+1+2(q_1-q_j)}} \leq \frac{1}{E^{a_j}} \frac{1}{A^{2(q_1-q_j)}} \frac{1}{K^{a_j+1+2(q_1-q_j-j)}} \leq \frac{1}{E} < 1.$$

(31), *third inequality*: It follows immediately from the 2nd inequality in (27) and from the second inequality in (31).

(32): $\frac{r_0}{\rho} = \left(4\sqrt{2} \frac{Mr}{\underline{\omega}} \mu_0 A^{q_1} K^{q_1-\frac{1}{\alpha_{n-1}}}\right)^{-1} < 1$ since $q_1 \geq 2 > 1/\alpha_{n-1}$, $K \geq 1$, and $\mu_0 \geq \frac{Mr}{\underline{\omega}} \frac{6\sqrt{2}}{n}$.

$\frac{r_{n-1}}{\rho} = K^{-\frac{q_{n-1}-1}{\alpha_{n-1}}} \frac{\kappa_{n-1}}{r\mu_0} \left(\frac{\underline{\omega}}{C_{n-1}} \frac{1}{2\sqrt{2}A^{q_{n-1}}}\right)^{\frac{1}{\alpha_{n-1}}} \leq \left(\frac{6E}{A}\right)^{\frac{1}{\alpha_{n-1}}} = 1$, where in the inequality we used $q_{n-1} \geq 2 > 1/\alpha_{n-1}$, $K \geq 1$, $\frac{1}{\mu_0} \leq \frac{r}{\kappa_{n-1}} \left(\frac{12\sqrt{2}EC_{n-1}}{\underline{\omega}}\right)^{\frac{1}{\alpha_{n-1}}}$ and $q_{n-1} \geq 1$.

Now, let $1 \leq j \leq n-2$ and define $\alpha'_j := \alpha_j - 1$, $b_j := \frac{q_j}{\beta_j} - 1$ (recall (6)) and observe that

$$b_j \geq \frac{\alpha_j}{\beta_j} > 0, \quad \frac{1}{\alpha_j} - 1 + \frac{q_j \rho'_j}{\alpha_j \beta_j} = \left(1 - \frac{1}{\alpha_j}\right) b_j. \quad (\text{B.4})$$

Observe also that from the definitions of κ_j , E and μ_0 it follows that

$$\kappa_j \geq \frac{\underline{\omega}}{M} \left(\frac{C_j}{\underline{\omega}}\right)^{\frac{1}{\alpha_j}} = \frac{\underline{\omega}^{1-\frac{1}{\alpha_j}} C_j^{\frac{1}{\alpha_j}}}{M}, \quad E \geq \left(\frac{(M\kappa_j)^{\alpha_j}}{C_j \underline{\omega}^{\alpha'_j}}\right)^{\frac{1}{\beta_j}}, \quad \mu_0 \geq \frac{\underline{\omega}}{r 24\sqrt{2} M}. \quad (\text{B.5})$$

Then, noticing also that $\frac{24\sqrt{2}}{6^{\frac{q_j}{\alpha_j}}} \leq \frac{24\sqrt{2}}{6^2} < 1$, we obtain

$$\begin{aligned} \frac{r_j}{\rho} &= K^{-\frac{q_j-1}{\alpha_{n-1}}} \frac{\kappa_j}{r\mu_0} \left(\frac{\underline{\omega}}{C_j} \frac{1}{2\sqrt{2}(6E)^{q_j}}\right)^{\frac{1}{\alpha_j}} \leq \frac{M}{\underline{\omega}} \kappa_j \left(\frac{\underline{\omega}}{C_j}\right)^{\frac{1}{\alpha_j}} \left(\frac{C_j \underline{\omega}^{\alpha'_j}}{(M\kappa_j)^{\alpha_j}}\right)^{\frac{q_j}{\alpha_j \beta_j}} \\ &\stackrel{(\text{B.4})}{=} \left(\frac{\underline{\omega}^{1-\frac{1}{\alpha_j}} C_j^{\frac{1}{\alpha_j}}}{M \kappa_j}\right)^{b_j} \stackrel{(\text{B.5})}{\leq} 1. \end{aligned}$$

(33): Using $q_1 + 1 = \frac{1}{2a}$, $6E \geq 1$ and $\varepsilon \leq \varepsilon_0 \leq \varepsilon_* \cdot \left(\min\left(1, \frac{6\sqrt{2}}{n} \frac{Mr}{\underline{\omega}}\right)\right)^{\frac{1}{b}}$ one finds:

$$\frac{r_0}{r} = \frac{\underline{\omega}}{4\sqrt{2}(6E)^{np_1-1}Mr} \left(\frac{\varepsilon}{\varepsilon_*}\right)^{\frac{1}{2}} \leq \frac{\underline{\omega}}{4\sqrt{2}Mr} \min\left(1, \frac{6\sqrt{2}}{n} \frac{Mr}{\underline{\omega}}\right) \leq 1.$$

(34): Since $\rho = R/(2n)$, (34) follows at one from (32).

(35): Set $x = \frac{n}{6\sqrt{2}} \frac{\underline{\omega}}{Mr}$, $y = \frac{n}{18\sqrt{2}}$, $z = \frac{4n\kappa_{n-1}}{r} \left(\frac{\underline{\omega}}{12\sqrt{2}EC_{n-1}}\right)^{\frac{1}{\alpha_{n-1}}}$. Then, from the definitions given and the hypothesis $\varepsilon \leq \varepsilon_0$ it follows:

$$\frac{2R}{r} = 4n\mu_0 \left(\frac{\varepsilon}{\varepsilon_*}\right)^b \leq 4n\mu_0 \left(\frac{\varepsilon_0}{\varepsilon_*}\right)^b = \max(x, y, z) \min(x^{-1}, y^{-1}, z^{-1}, 1) \leq 1.$$

(36): From the definitions given (and since $\frac{e}{24} > \frac{1}{10}$) it follows:

$$\begin{aligned} T_{\text{exp}} \frac{\sqrt{\varepsilon}}{T e^{\frac{Ks}{6}}} &= 6A^{\frac{1}{a}} \sqrt{\frac{\varepsilon_*}{\varepsilon}} \min\left(\frac{1}{10} \frac{1}{(AK)^{q_1}} \frac{1}{K}, \frac{e}{24} \min_{\substack{1 \leq j \leq n-1 \\ \Lambda: \dim \Lambda = j}} \left(\frac{1}{|\Lambda|} \frac{1}{(AK)^{q_j}}\right)\right) \\ &\geq \frac{3}{5} A^{\frac{1}{a}} \sqrt{\frac{\varepsilon_*}{\varepsilon}} \min_{1 \leq j \leq n-1} \frac{1}{A^{q_j}} \frac{1}{K^{q_j+1}} = \frac{3}{5} K A^{np_1+1} \geq \frac{3}{5} A^4 > 1. \end{aligned}$$

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